THE NONCOMMUTATIVE GEOMETRY OF k-GRAPH C^*-ALGEBRAS

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ABSTRACT. This paper is comprised of two related parts. First we discuss which k-graph algebras have faithful traces. We characterise the existence of a faithful semifinite lower-semicontinuous gauge-invariant trace on $C^*(\Lambda)$ in terms of the existence of a faithful graph trace on Λ.

Second, for k-graphs with faithful gauge invariant trace, we construct a smooth (k, ∞)-summable semifinite spectral triple. We use the semifinite local index theorem to compute the pairing with K-theory. This numerical pairing can be obtained by applying the trace to a KK-pairing with values in the K-theory of the fixed point algebra of the $\mathbf{T}^k$ action. As with graph algebras, the index pairing is an invariant for a finer structure than the isomorphism class of the algebra.

1. INTRODUCTION

In this paper we generalise the construction of semifinite spectral triples for graph algebras of [PRen] to the C^*-algebras of higher-rank graphs, or k-graphs. Experience with k-graph algebras has shown that from a C^*-algebraic point of view they tend to behave very much like graph C^*-algebras. Consequently the transition from graph C^*-algebras to k-graph C^*-algebras often appears quite simple. The subtlety generally lies in the added combinatorial complexity of k-graphs, and in particular in identifying the right higher-dimensional analogues of the graph-theoretic conditions which arise in the one-dimensional case. This experience is borne out again in the current paper: once the appropriate k-graph theoretic conditions have been identified, the generalisations of the constructions in [PRen] to higher-rank graphs turn out to be mostly straightforward.

However, the pay-offs from carrying through this analysis are significant from the points of view of both noncommutative geometry and k-graph algebra theory.

The pay-off for noncommutative geometry is that our analysis allows us to construct infinitely many examples of (semifinite) spectral triples of every integer dimension $k \geq 1$ ($k = 1$ is contained in [PRen]). These spectral triples are generically semifinite, and so come from KK classes rather than K-homology classes. Computations can be made very explicitly with these algebras, and we use this to relate the semifinite index pairing to the KK-index for these

examples. This has led to a general picture of the relationship between semifinite index theory and KK-theory, [KNR]. From our point of view, k-graph algebras are sufficiently generic to reveal the relationship between KK-theory and semifinite index theory. We highlight this at the end of Section 7.

The connections between non-commutative geometry and classical (commutative) differential geometry are still a subject of intense research (and speculation). This paper helps to illuminate these connections in two ways. The first, and always the most important for understanding the extension of geometry to the noncommutative setting, is index theory. We relate an analytic index to a KK-index with values in the K-theory of the fixed point algebra. This KK-index is far better adapted to topological interpretation, and furthermore exists in greater generality.

The second connection between this paper and classical geometry is less deep, but is a key tool in our construction. The group $\mathbf{T}^k$ acting on a k-graph algebra is a Lie group. We use the action of $\mathbf{T}^k$ to 'push forward' the Dirac operator on $\mathbf{T}^k$ to a Dirac operator for a k-graph algebra. While we have done this only for the (spin) Dirac operator of the simplest spin structure on $\mathbf{T}^k$, the possibility also exists for repeating our construction for every spin^c structure on $\mathbf{T}^k$, as well as the Hodge-de Rham operator, and other twisted Dirac operators. It is an interesting question to determine to what extent this 'pushing forward' operation can be systematised.

The pay-off for k-graph algebra theory is that we obtain detailed information about semifinite traces on k-graph algebras. To construct a semifinite spectral triple one requires a faithful semifinite trace on the underlying C^*-algebra. Hence our first step is to investigate when such a trace exists on a k-graph algebra. We characterise faithful semifinite gauge-invariant traces on $C^*(\Lambda)$ in terms of *graph traces* on Λ (cf. [H, PRen]). In particular, this represents a first systematic exploration of gauge-invariant traces on C^*-algebras associated to graphs of arbitrary rank (see Section 3).

In the appendix, we also identify a (fairly restrictive) class of k-graphs which admit faithful graph traces, demonstrate that their C^*-algebras are Morita equivalent to direct sums of algebras of continuous functions on tori of rank $0, \ldots, k$, and calculate their K-theory. For graph algebras this is not new because the K-theory of graph algebras is completely understood [RSz]. However, only for 2-graphs have general K-theory computations recently emerged [E]. Consequently any advances on K-theory for general k-graph algebras are significant.

Outline. The paper is arranged a follows. In Section 2 we review the basic definitions of k-graphs and k-graph algebras. In Section 3 we show that a k-graph algebra $C^*(\Lambda)$ admits a faithful, semifinite, lower-semicontinuous, gauge-invariant trace if and only if Λ admits a faithful graph trace (in the Appendix, we identify a substantial class of k-graphs which admit such a graph trace, show that they are Morita equivalent to commutative C^*-algebras, and hence compute their K-theory).

Section 4 reviews the definitions we require pertaining to semifinite spectral triples. In Section 5 we construct a Kasparov module for the C^*-algebra of any locally finite, locally convex k-graph with no sinks. This is a very general construction, and the resulting Kasparov module is even

iff k is an even integer. In Section 6 we construct (k, ∞)-summable spectral triples for k-graph algebras with faithful trace.

The von Neumann algebra constructed as part of our spectral triple plays the role of the crossed product of the (von Neumann completion of the) graph algebra by the $\mathbf{T}^k$ action. The precise relationship is not clear to us, but we suspect the two are isomorphic.

In Section 7 we use these spectral triples to compute index pairings and compare them with the Kasparov product as in [PRen]. By example we indicate how the semifinite index can be used to obtain more refined information than the usual Fredholm index.

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2. k-GRAPH C^*-ALGEBRAS

2.1. **Higher-rank graphs and their C^*-algebras.** In this subsection we outline the basic notation and definitions of k-graphs and their C^*-algebras. We refer the reader to [RSY] for a more thorough account.

Higher-rank graphs. Throughout this paper, we regard $\mathbf{N}^k$ as a monoid under pointwise addition. We denote the usual generators of $\mathbf{N}^k$ by $e_1, \ldots, e_k$, and for $n \in \mathbf{N}^k$ and $1 \leq i \leq k$, we denote the ith coordinate of n by $n_i \in \mathbf{N}$; so $n = \sum n_i e_i$. For $m, n \in \mathbf{N}^k$, we write $m \leq n$ if $m_i \leq n_i$ for all i. By $m < n$, we mean $m \leq n$ and $m \neq n$. We use $m \vee n$ and $m \wedge n$ to denote, respectively, the coordinate-wise maximum and coordinate-wise minimum of m and n; so that $m \wedge n \leq m, n \leq m \vee n$ and these are respectively the greatest lower bound and least upper bound of m, n in $\mathbf{N}^k$.

Definition 2.1 (Kumjian-Pask [KP]). *A graph of rank k or k-graph is a pair (Λ, d) consisting of a countable category Λ and a* degree *functor $d : \Lambda \to \mathbf{N}^k$ which satisfy the following* factorisation property*: if $\lambda \in \text{Mor}(\Lambda)$ satisfies $d(\lambda) = m + n$, then there are unique morphisms $\mu, \nu \in \text{Mor}(\Lambda)$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu \circ \nu$.*

The factorisation property ensures (see [KP]) that the identity morphisms of Λ are precisely the morphisms of degree 0; that is $\{\text{id}_o : o \in \text{Obj}(\Lambda)\} = d^{-1}(0)$. This means that we may identify each object with its identity morphism, and we do this henceforth. This done, we can regard Λ as consisting only of its morphisms, and we write $\lambda \in \Lambda$ to mean $\lambda \in \text{Mor}(\Lambda)$.

Since we are thinking of Λ as a kind of graph, we write r and s for the codomain and domain maps of Λ respectively. We refer to elements of Λ as *paths*, and to the paths of degree 0 (which correspond to the objects of Λ as above) as *vertices*. Extending these conventions, we refer to the elements of Λ with minimal nonzero degree (that is $d^{-1}(\{e_1, \ldots, e_k\})$) as *edges*.

Notation 2.2. To try to minimise confusion, we will always use u, v, w to denote vertices, e, f to denote edges, and lower-case Greek letters λ, μ, ν, etc. for arbitrary paths. We will also drop the composition symbol, and simply write $\mu\nu$ for $\mu \circ \nu$ when the two are composable.

WARNING: *because Λ is a category, composition of morphisms reads from right to left. Hence paths μ and ν in Λ can be composed to form $\mu\nu$ if and only if $r(\nu) = s(\mu)$, and in this case, $r(\mu\nu) = r(\mu)$ and $s(\mu\nu) = s(\nu)$. This is the reverse of the convention for directed graphs, used in [BPRS, KPR, KPRR, PRen], so the reader should beware. In particular the roles of sources and sinks, and of ranges and sources, are opposite to those in [PRen].*

Definition 2.3. *For each $n \in \mathbf{N}^k$, we write Λ^n for the collection $\{\lambda \in \Lambda : d(\lambda) = n\}$ of paths of degree n.*

The range and source r, s are thus maps from Λ to Λ^0, and if $v \in \Lambda^0$, then $r(v) = v = s(v)$.

Given $\lambda \in \Lambda$ and $S \subset \Lambda$, it makes sense to write λS for $\{\lambda\sigma : \sigma \in S, r(\sigma) = s(\lambda)\}$, and likewise $S\lambda = \{\sigma\lambda : \sigma \in S, s(\sigma) = r(\lambda)\}$. In particular, if $v \in \Lambda^0$, then vS is the collection of elements of S with range v, and Sv is the collection of elements of S with source v.

Definition 2.4. *Let (Λ, d) be a k-graph. We say that Λ is* row-finite *if $|v\Lambda^n| < \infty$ for each $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We say that Λ is* locally-finite *if it is row-finite and also satisfies $|\Lambda^n v| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We say that Λ has* no sources *(resp. no sinks) if $v\Lambda^n$ (resp. $\Lambda^n v$) is nonempty for each $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. Finally, we say that Λ is* locally convex *if, for each edge $e \in \Lambda^{e_i}$, and each $j \neq i$ in $\{1, \ldots, k\}$, we have $s(e)\Lambda^{e_j} = \emptyset$ only if $r(e)\Lambda^{e_j} = \emptyset$.*

As in [RSY], for locally convex k-graphs, we use the notation $\Lambda^{\leq n}$ to denote the collection

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n, \mu \in s(\lambda)\Lambda \text{ and } d(\lambda\mu) \leq n \text{ implies } \mu = s(\lambda)\}.$$

Intuitively, $\Lambda^{\leq n}$ is the collection of paths whose degree is "as large as possible" subject to being dominated by n. In a 1-graph, $\Lambda^{\leq n}$ is the set of paths $\lambda \in \Lambda$ whose length is at most n and is less than n only if $s(\lambda)$ receives no edges. The significance of this is that the partial isometries associated to distinct paths in $\Lambda^{\leq n}$ have orthogonal range projections (cf. relation (CK4) below). For more on the importance of $\Lambda^{\leq n}$, see [RSY].

$\Omega_{\mathbf{k,m}}$ and boundary paths. For $k \geq 1$ and $m \in (\mathbf{N} \cup \{\infty\})^k$, we define a k-graph $\Omega_{k,m}$ as follows:

$$\Omega_{k,m}^0 = \{n \in \mathbf{N}^k : n \leq m\} \qquad \Omega_{k,m}^n = \{(p, q) \in \mathbf{N}^k : p, q \in \Omega_{k,m}^0, q - p = n\}$$
$$r(p, q) = p, \qquad s(p, q) = q, \qquad (p, q) \circ (q, n) = (p, n).$$

See Figure 1 for a "picture" of $\Omega_{3,(\infty,2,1)}$.

Each path λ of degree p in a k-graph Λ determines a degree-preserving functor $\hat{\lambda}$ from $\Omega_{k,p}$ to Λ as follows: the image $\hat{\lambda}(m, n)$ of the morphism $(m, n) \in \Omega_{k,p}$ is the unique morphism in Λ^{n-m} such that there exist $\mu \in \Lambda^m$ and $\nu \in \Lambda^{p-n}$ satisfying $\lambda = \mu\hat{\lambda}(m, n)\nu$. (The existence and uniqueness of $\hat{\lambda}(m, n)$ is guaranteed by the factorisation property).

In fact for each $p \in \mathbf{N}^k$, the map $\lambda \mapsto \hat{\lambda}$ is a bijection between Λ^p and the set of degree-preserving functors from $\Omega_{k,p}$ to Λ. In practise, we just write $\lambda(m,n)$ for the segment $\hat{\lambda}(m,n)$ of λ described in the previous paragraph, and we write $\lambda(m)$ for the range of the path $\lambda(m,n)$, which we think of as the vertex on λ at position m. If $\lambda \in \Lambda^p$ and $0 \le m \le n \le p$, then

$$\lambda = \lambda(0,m)\lambda(m,n)\lambda(n,p), \quad s(\lambda(m,n)) = \lambda(n) \quad \text{and} \quad r(\lambda(m,n)) = \lambda(m).$$

We extend this correspondence between paths and degree-preserving functors to define the notion of a boundary path in a k-graph.

Definition 2.5. *A* boundary path *of a k-graph Λ is a degree-preserving functor $x : \Omega_{k,m} \to \Lambda$ such that*

$$\text{if } m_i < \infty, \ n \in \mathbf{N}^k, \ n \le m \text{ and } n_i = m_i, \text{ then } x(n)\Lambda^{e_i} = \emptyset;$$

so the directions in which x is finite are those in which it cannot be extended. If $x : \Omega_{k,m} \to \Lambda$ is a boundary path, we denote m by $d(x)$, and $x(0)$ by $r(x)$. We write $\Lambda^{\le\infty}$ for the set of all boundary paths of Λ.

Note that if $\lambda \in \Lambda$ satisfies $s(\lambda)\Lambda^n = \emptyset$ for all $n > 0$ (that is, if $s(\lambda)$ is a source in Λ), then the graph morphism $\hat{\lambda} : \Omega_{k,d(\lambda)} \to \Lambda$ discussed above belongs to $\Lambda^{\le\infty}$; we think of λ itself as a boundary path of Λ.

Definition 2.6. *An* end *of Λ is a boundary path $x \in \Lambda^{\le\infty}$ such that for all $n \le d(x)$, $r(x)\Lambda^n = \{x(0,n)\}$. We denote the set of ends of Λ by $Ends(\Lambda)$.*

Remarks 2.7. If x is an end of Λ, then $r(x)\Lambda^{\le n} = \{x(0, n \wedge d(x))\}$ for all $n \in \mathbf{N}^k$.

Skeletons. To draw a k-graph, we use its *skeleton*. The skeleton of a k-graph Λ is the directed graph whose vertices and edges are those of Λ, but with the k different types of edges distinguished using k different colours. In this paper, we use solid lines for edges of degree e_1, dashed lines for edges of degree e_2, and dotted lines for edges of degree e_3. For example, the skeleton of $\Omega_{3,(\infty,2,1)}$ is presented in Figure 1

The factorisation property says that if e and f are edges of degree e_i and e_j respectively such that $s(e) = r(f)$, then the path ef can be expressed in the form $f'e'$ where $d(f') = e_j$ and $d(e') = e_i$. In the skeleton for $\Omega_{3,(\infty,2,1)}$ there is just one way this can happen; so the skeleton is actually a commuting diagram in the category, and although there appear to be many ways to get from $(1,2,1)$ to $(0,0,0)$, for example, each of these paths yields the same morphism in the category, so there is really just one path in $\Omega_{3,(\infty,2,1)}$ from $(1,2,1)$ to $(0,0,0)$.

The information determining the factorisation property is not always included in the skeleton, and it must then be specified separately as a set of *factorisation rules*. The uniqueness of factorisations ensures that amongst the factorisation rules for the skeleton of a k-graph, each composition ef where e and f are composable edges of different colours will appear exactly once. A set of factorisation rules for a skeleton with this property is referred to as an *allowable* factorisation regime.
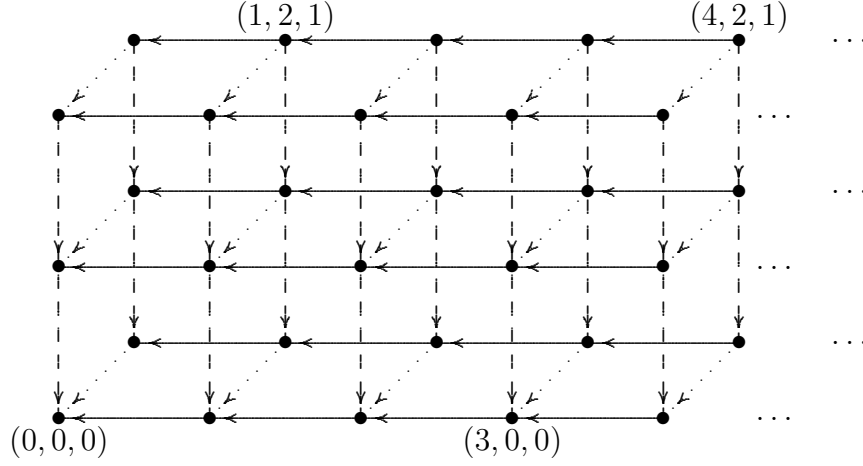
FIGURE 1. The skeleton of $\Omega_{3,(\infty,2,1)}$

For example, in the 1-skeleton of Figure 2 the allowable factorisation regimes are: $\{ef = he, kf = hk\}$ and $\{ef = hk, kf = he\}$).
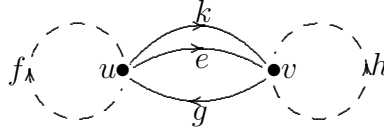

FIGURE 2. A skeleton that admits two distinct factorisation regimes

A skeleton together with an allowable factorisation regime determines at most one k-graph. When $k = 2$, each skeleton and allowable factorisation regime determines a unique k-graph. For $k \geq 3$, there is an additional associativity condition on the factorisation rules which must be verified [FS]; but the issue does not arise in the examples we give in this paper.

Cuntz-Krieger families and $C^*(\Lambda)$. As with directed graphs, we are interested in higher-rank graphs because we can associate to each one a C^*-algebra of Cuntz-Krieger type.

Definition 2.8. *Let (Λ, d) be a row-finite locally convex k-graph. A Cuntz-Krieger Λ-family is a collection $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying*

(CK1) $\{s_v : v \in \Lambda^0\}$ *is a collection of mutually orthogonal projections;*
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ *for all $\mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;*
(CK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ *for all $\lambda \in \Lambda$; and*
(CK4) $s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$ *for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$.*

As a point of notation, we will henceforth denote the vertex projection s_v by p_v to remind ourselves that it is a projection.

The Cuntz-Krieger algebra of Λ, denoted $C^*(\Lambda)$, is the universal C^*-algebra generated by a Cuntz-Krieger family $\{s_\lambda : \lambda \in \Lambda\}$. By this we mean that given any other Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$, there is a homomorphism π_t satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. By [RSY, Proposition 3.5], if $\mu, \nu \in \Lambda$, then $s_\mu^* s_\nu = \sum_{\mu\alpha=\nu\beta,d(\mu\alpha)=d(\mu)\vee d(\nu)} s_\alpha s_\beta^*$, and hence ([RSY, Remarks 3.8(1)]),

$$\text{(1)} \qquad\qquad C^*(\Lambda) = \overline{\text{span}}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta)\}.$$

For the details of the next two paragraphs, see [RSY, page 109].

The universal property of $C^*(\Lambda)$ guarantees that there is an action $\gamma : \mathbf{T}^k \to \text{Aut}(C^*(\Lambda))$ satisfying $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda := z_1^{d(\lambda)_1} \cdot z_2^{d(\lambda)_2} \cdots z_k^{d(\lambda)_k} s_\lambda$ and hence $\gamma_z(p_v) = p_v$. We denote the fixed point algebra for γ by F, and Φ denotes the faithful conditional expectation $\Phi : C^*(\Lambda) \to F$ determined by $\Phi(a) = \int_{\mathbf{T}^k} \gamma_z(a)\, d\mu(z)$.

We have $F = \overline{\text{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu), s(\mu) = s(\nu)\}$ and Φ is determined by $\Phi(s_\mu s_\nu^*) = \delta_{d(\mu),d(\nu)} s_\mu s_\nu^*$. For each $n \in \mathbf{N}^k$, we write $F_n := \overline{\text{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu), \mu, \nu \in \Lambda^{\leq n}, s(\mu) = s(\nu)\}$. Then each F_n is isomorphic to a direct sum of matrix algebras and algebras of compact operators, and $F = \overline{\bigcup F_n}$ is an AF algebra.

3. k-GRAPH TRACES AND FAITHFUL TRACES ON $C^*(\Lambda)$

In this section we investigate conditions which give rise to faithful traces on $C^*(\Lambda)$ for a locally convex locally finite k-graph Λ. As with the C^*-algebras of directed graphs, necessary and sufficient conditions for the existence of faithful traces on a k-graph algebra are hard to come by. We denote by A^+ the positive cone in a C^*-algebra A, and we use extended arithmetic on $[0, \infty]$ so that $0 \times \infty = 0$. From [PhR] we take the basic definition:

Definition 3.1. *A trace on a C^*-algebra A is a map $\tau : A^+ \to [0, \infty]$ satisfying*

1) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in A^+$

2) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A^+$ and $\lambda \geq 0$

3) $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$

We say: that τ is faithful if $\tau(a^*a) = 0 \Rightarrow a = 0$; that τ is semifinite if $\{a \in A^+ : \tau(a) < \infty\}$ is norm dense in A^+ (or that τ is densely defined); that τ is lower semicontinuous if whenever $a = \lim_{n\to\infty} a_n$ in norm in A^+ we have $\tau(a) \leq \liminf_{n\to\infty} \tau(a_n)$.

We may extend a (semifinite) trace τ by linearity to a linear functional on (a dense subspace of) A. Observe that the domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$. The proof of the following Lemma is identical to that of the analogous result for graph algebras [PRen, Lemma 3.2].

Lemma 3.2. *Let (Λ, d) be a row-finite locally convex k-graph and let $\tau : C^*(\Lambda) \to \mathbf{C}$ be a semifinite trace. Then the dense subalgebra*

$$A_c := \mathrm{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$$

is contained in the domain I_τ of τ.

Recall from [Si] that a loop with an entrance is a path $\lambda \in \Lambda$ with $r(\lambda) = s(\lambda)$ such that $d(\lambda) \geq e_i$ for some $1 \leq i \leq k$, together with an $e \in \Lambda^{e_i}$ with $r(e) = r(\lambda)$ but $\lambda(0, e_i) \neq e$.

Lemma 3.3. *Let (Λ, d) be a row-finite locally convex k-graph.*

(i) *If $C^*(\Lambda)$ has a faithful semifinite trace then no loop can have an entrance.*

(ii) *If $C^*(\Lambda)$ has a gauge-invariant, semifinite, lower semicontinuous trace τ then $\tau \circ \Phi = \tau$ and*

$$\tau(s_\mu s_\nu^*) = \delta_{\mu,\nu} \tau(p_{s(\mu)}).$$

In particular, if $\tau|_{C^*(\{s_\mu s_\mu^* : \mu \in \Lambda\})} = 0$ then $\tau = 0$.

Proof. The entrance condition implies that $\lambda(0, e_i)$ and the entrance e are distinct paths of degree e_i with the same range, and it follows from (CK3) and (CK4) that

$$s_\lambda^* s_\lambda = p_{s(\lambda)} = p_{r(\lambda)} \geq s_\lambda s_\lambda^* + s_e s_e^*.$$

If τ is a trace on $C^*(\Lambda)$, we therefore have $\tau(s_\lambda^* s_\lambda) \geq \tau(s_\lambda s_\lambda^*) + \tau(s_e s_e^*)$, and it follows from Lemma 3.2 and the trace property that $\tau(s_e^* s_e) = \tau(s_e s_e^*) = 0$. Theorem 3.15 of [RSY] implies that $s_e^* s_e \neq 0$ so τ is not faithful.

The proof of the second part is the same as [PRen, Lemma 3.3], but for clarity we remind the reader how the final statement arises. If τ is gauge invariant we have

$$\tau(s_\mu s_\nu^*) = \tau(\gamma_z(s_\mu s_\nu^*)) = z^{d(\mu)-d(\nu)} \tau(s_\mu s_\nu^*)$$

for all $z \in \mathbf{T}^k$. Hence $\tau(s_\mu s_\nu^*)$ is zero unless $d(\mu) = d(\nu)$, and so $\tau = \tau \circ \Phi$. Moreover if $d(\mu) = d(\nu)$, then using the trace property,

$$\tau(s_\mu s_\nu^*) = \tau(s_\nu^* s_\mu) = \delta_{\nu,\mu} \tau(p_{s(\nu)}) = \delta_{\nu,\mu} \tau(s_\nu^* s_\nu).$$

This proves that if $\tau|_{\mathrm{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}} = 0$ then $\tau|A_c = 0$. The details of extending this to the C^*-completion are as in [PRen]. □

Whilst the condition that no loop has an entrance is necessary for the existence of a faithful semifinite trace, it is not sufficient. For example, let Λ be any 2-graph whose skeleton is the one illustrated in Figure 3 (there are many allowable factorisation regimes to choose from). Then Λ is locally convex and locally finite, contains no sinks or sources, and contains no cycles at all, so certainly no cycles with entrances, yet $C^*(\Lambda)$ does not admit a faithful semifinite trace. To see why note that (CK4) forces $s_g s_g^* = p_v = s_e s_e^* + s_f s_f^*$ so if τ is a trace on $C^*(\Lambda)$ then the trace property forces

$$\tau(p_v) = \tau(s_g s_g^*) = \tau(p_w) \quad \text{and} \quad \tau(p_v) = \tau(s_e s_e^*) + \tau(s_f s_f^*) = 2\tau(p_w),$$

FIGURE 3. A 2-graph whose C^*-algebra does not admit a faithful semifinite trace

and hence $\tau(p_w) = 0$.

The situation illustrated in Figure 3 is more subtle than those which can arise for graph C^*-algebras. However, as with directed graphs, the obstructions to the existence of a faithful semifinite trace on a k-graph algebra can be expressed most naturally for general k-graphs Λ in terms of a function $g_\tau : \Lambda^0 \to \mathbf{R}^+$ which arises naturally from each trace τ on $C^*(\Lambda)$.

Lemma 3.4. *Let Λ be a locally convex row-finite k-graph, and suppose that τ is a semifinite trace on $C^*(\Lambda)$. Then the function $g_\tau : \Lambda^0 \to \mathbf{R}^+$ defined by $g_\tau(v) := \tau(p_v)$ satisfies $g_\tau(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g_\tau(s(\lambda))$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$.*

Proof. Fix $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. By (CK4), we have $p_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$. Hence

$$\tau(p_v) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(s_\lambda s_\lambda^*) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(s_\lambda^* s_\lambda) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(p_{s(\lambda)}),$$

and the result follows from the definition of g_τ. $\qquad\square$

Returning to the example of Figure 3 we can see that if τ is a trace on $C^*(\Lambda)$ then $g_\tau(v)$ must simultaneously be equal to $g_\tau(w)$ and $2g_\tau(w)$, forcing $g_\tau(w)$ and hence $\tau(p_w)$ to be equal to zero.

Motivated by Lemma 3.4, we make the following definition (see [H] for the origins of this definition):

Definition 3.5. *Let Λ be a locally convex row-finite k-graph. A function $g : \Lambda^0 \to \mathbf{R}^+$ is called a k-graph trace on Λ if it satisfies*

$$(2) \qquad g(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda)) \text{ for all } v \in \Lambda^0 \text{ and } n \in \mathbf{N}^k.$$

We say that g is faithful *if $g(v) \neq 0$ for all $v \in \Lambda^0$.*

Remarks 3.6. Notice that if x is an end of Λ, then $x(0)\Lambda^{\leq n} = \{x(0, n)\}$ for any $n \leq d(x)$. It follows that each k-graph trace on Λ is constant on the vertices of x.

We want to be able to construct semifinite lower semicontinuous gauge-invariant traces on $C^*(\Lambda)$ from k-graph traces on Λ. The idea is to use (2) to define a trace on $C^*(\Lambda)$ by $\tau_g\left(\sum_{\mu,\nu \in F} a_{\mu,\nu} s_\mu s_\nu^*\right) = \sum_{\mu \in F} a_{\mu,\mu} g(s(\mu))$. There are two problems to overcome: is τ_g well-defined in the first place, and when is τ_g faithful? To address these problems, we establish that there is a faithful conditional expectation Ψ on $C^*(\Lambda)$ satisfying $\Psi(s_\mu s_\nu^*) = \delta_{\mu,\nu} s_\mu s_\mu^*$. We would

like to thank the referee for pointing out the straightforward proof of this result appearing below.

Proposition 3.7. *Let Λ be a locally convex row-finite k-graph. There is a faithful conditional expectation $\Psi : C^*(\Lambda) \to D := \overline{\mathrm{span}}\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}$ which satisfies $\Psi(s_\lambda s_\mu^*) = \delta_{\lambda,\mu} s_\lambda s_\lambda^*$ for all $\lambda, \mu \in \Lambda$.*

Proof. Averaging over the gauge action γ gives a faithful conditional expectation Φ^γ onto the fixed point algebra $C^*(\Lambda)^\gamma = \overline{\mathrm{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$. For $q \in \mathbf{N}^k$, $p \leq q$ and $v \in \Lambda^0$, let $P(q,p,v) := \{\lambda \in \Lambda^{\leq q} : s(\lambda) = v, d(\lambda) = p\}$. It is shown on page 109 of [RSY] that $\mathcal{F}_{q,p}(v) := \overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu \in P(q,p,v)\}$, is canonically isomorphic to $\mathcal{K}(\ell^2(P(q,p,v)))$, that for fixed q, distinct $\mathcal{F}_{q,p}(v)$ are orthogonal, and that setting $\mathcal{F}_q := \bigoplus_{p,v} \mathcal{F}_{q,p}(v)$, we have

$$C^*(\Lambda)^\gamma = \overline{\bigcup_{q \in \mathbf{N}^k} \mathcal{F}_q}$$

This shows that $C^*(\Lambda)^\gamma$ is an AF algebra with maximal abelian subalgebra D. Let Φ^D denote the canonical expectation from $C^*(\Lambda)^\gamma$ onto its maximal abelian subalgebra. Then $\Psi := \Phi^D \circ \Phi^\gamma$ is the desired expectation. $\square$

Proposition 3.8. *Let Λ be a row-finite locally convex k-graph. Then there is a one-to-one correspondence between faithful graph traces on Λ and faithful, semifinite, lower semicontinuous, gauge invariant traces on $C^*(\Lambda)$.*

Proof. Given a faithful k-graph trace, the existence of $\Psi : C^*(\Lambda) \to \overline{\mathrm{span}}\{S_\mu S_\mu^*\}$ given by Proposition 3.7 shows that the functional $\tau_g : A_c \to \mathbf{C}$ defined by

$$\tau_g(S_\mu S_\nu^*) := \delta_{\mu,\nu} g(s(\mu))$$

is well-defined. As in [PRen], one checks that τ_g is a gauge invariant trace on A_c and is faithful because for $a = \sum_{i=1}^n c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^* \in A_c$ we have $a^* a \geq \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 S_{\nu_i} S_{\nu_i}^*$, so

(3)
$$\langle a, a \rangle_g := \tau_g(a^* a) = \tau_g\Big(\sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 S_{\nu_i} S_{\nu_i}^* \Big) = \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 \tau_g(S_{\nu_i} S_{\nu_i}^*) = \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 g(s(\nu_i)) > 0.$$

by definition of τ_g.

Then $\langle a, b \rangle_g = \tau_g(b^* a)$ defines a positive definite inner product on A_c which makes it a Hilbert algebra (that the left regular representation of A_c is nondegenerate follows from $A_c^2 = A_c$). The rest of the proof is the same as [PRen, Proposition 3.9], except that we use the gauge invariant uniqueness theorem for k-graphs, [RSY, Theorem 4.1], to show that we obtain faithful representation of $A = C^*(\Lambda)$ on the Hilbert space completion of A_c. $\square$

The results of sections 4–7 below apply to any k-graph which admits a faithful graph trace. It is therefore important to establish that there is a substantial class of k-graphs for any k with this property, and how large the class is. This is in general a difficult question. The results of [PRRS] show there is a substantial class of 2-graphs admitting faithful graph traces, but for

arbitrary k there are, to our knowledge, no definitive results. In Appendix A we establish some necessary conditions and one sufficient condition, which will prove useful in section 7.

4. Semifinite Spectral Triples

We begin with some semifinite versions of standard definitions and results. Let τ be a fixed faithful, normal, semifinite trace on the von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_\mathcal{N}$ be the τ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

Definition 4.1. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(\mathcal{N}, \tau)$ consists of a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$

2) $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_\mathcal{N}$ for all $\lambda \notin \mathbf{R}$ and all $a \in \mathcal{A}$

3) The triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$, and $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$. Otherwise it is odd.

Definition 4.2. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$.*

Note. The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{N}$ which need not be (weakly) dense in $\mathcal{N}$. More precisely, dom $\delta = \{T \in \mathcal{N} : \delta(T) \text{ is bounded}\}$. We also note that if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1 + \mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{N}$. In fact, writing $|\mathcal{D}|_1 = (1 + \mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have

$$\text{dom } \delta^n = \text{dom } \delta_1^n \quad \text{for all } n.$$

We also observe that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|, T]$, $[(1 + \mathcal{D}^2)^{1/2}, T]$. The proofs of these statements can be found in [CPRS2].

Definition 4.3. *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.*

Thus saying that $\mathcal{A}$ is *smooth* means that $\mathcal{A}$ is Fréchet and a pre-C^*-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A (since Fréchet means locally convex, metrizable and complete.) We will sometimes speak of $\overline{\mathcal{A}} = A$, particularly when $\mathcal{A}$ is represented on Hilbert space and

the norm closure $\overline{\mathcal{A}}$ is unambiguous. At other times we regard $i : \mathcal{A} \hookrightarrow A$ as an embedding of $\mathcal{A}$ in a C^*-algebra. We will use both points of view.

It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [GVF, S]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i. This definition ensures that a smooth algebra is a 'good' algebra, [GVF], so these algebras have a sensible spectral theory which agrees with that defined using the C^*-closure, and the group of invertibles is open.

Stability under the holomorphic functional calculus extends to nonunital algebras, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the 'one-point' unitization, though we must of course restrict to functions satisfying $f(0) = 0$. Likewise, the definition of a Fréchet algebra does not require a unit. The point of contact between smooth algebras and QC^∞ spectral triples is the following Lemma, proved in [R1].

Lemma 4.4. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_{n,i}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0, \ i = 0, 1,$$

where $d(a) = [\mathcal{D}, a]$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

We call the topology on $\mathcal{A}$ determined by the seminorms q_{ni} of Lemma 4.4 the δ-topology.

Whilst smoothness does not depend on whether $\mathcal{A}$ is unital or not, many analytical problems arise because of the lack of a unit. As in [R1, R2, GGISV], we make two definitions to address these issues.

Definition 4.5. *An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each i*

$$\phi a_i = a_i \phi = a_i.$$

Definition 4.6. *Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subseteq \mathcal{A}$ be a dense subalgebra with local units. Then we call $\mathcal{A}$ a quasi-local algebra (when $\mathcal{A}_c$ is understood.) If $\mathcal{A}_c$ is a dense ideal with local units, we call $\mathcal{A}_c \subset \mathcal{A}$ local.*

Quasi-local algebras have an approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n$, [R1].

Example For a k-graph C^*-algebra $A = C^*(\Lambda)$, Equation (1) shows that

$$A_c = \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } s(\mu) = s(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & v = r(\mu) \\ 0 & \text{otherwise} \end{cases}.$$

Similar comments apply to right multiplication by $p_{r(\nu)}$. By summing the source and range projections (without repetitions) of all $S_{\mu_i} S_{\nu_i}^*$ appearing in a finite sum

$$a = \sum_i c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for $a \in A_c$. By repeating this process for any finite collection of such $a \in A_c$ we see that A_c has local units.

We also require that when we have a spectral triple the operator $\mathcal{D}$ is compatible with the quasi-local structure of the algebra, in the following sense.

Definition 4.7. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega_{\mathcal{D}}^*(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.*

Definition 4.8. *A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ quasi-local such that there exists an approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying*

$$\Omega_{\mathcal{D}}^*(\mathcal{A}_c) = \bigcup_n \Omega_{\mathcal{D}}^*(\mathcal{A})_n,$$

$$\Omega_{\mathcal{D}}^*(\mathcal{A})_n = \{\omega \in \Omega_{\mathcal{D}}^*(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}.$$

Remark A local spectral triple has a local approximate unit $\{\phi_n\}_{n \geq 1} \subset \mathcal{A}_c$ such that, [R2], $\phi_{n+1}\phi_n = \phi_n \phi_{n+1} = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$. This is the crucial property we require to prove our summability results for nonunital spectral triples, to which we now turn.

4.1. **Summability.** In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace τ. Recall from [FK] that if $S \in \mathcal{N}$, the tth *generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{\|SE\| \; : \; E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\| \cdot \|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ is not complete in this norm but it is complete in the norm $\| \cdot \|_1 + \| \cdot \|_\infty$. (where $\| \cdot \|_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier trace:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \; : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

There are related ideals for $p > 1$: to describe them first set

$$\psi_p(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ t^{1-\frac{1}{p}} & \text{for } 1 \leq t. \end{cases}$$

Then define

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \|T\|_{\mathcal{L}^{(p,\infty)}} := \sup_{t>0} \frac{1}{\psi_p(t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

For $p > 1$ there is also the equivalent definition

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \sup_{t>0} \frac{t}{\psi_p(t)} \mu_t(T) < \infty \right\}.$$

If $T \in \mathcal{L}^{(p,\infty)}(\mathcal{N})$, then $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{N})$.

We will suppress the $(\mathcal{N})$ in our notation for these ideals, as $\mathcal{N}$ will always be clear from context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$. Our notation is however consistent with that of [C] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0.$$

Definition 4.9. *A semifinite local spectral triple is (k,∞)-summable if*

$$a(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(k,\infty)} \quad \text{for all } a \in \mathcal{A}_c, \ \ \lambda \in \mathbf{C} \setminus \mathbf{R}.$$

Remark If $\mathcal{A}$ is unital, $\ker \mathcal{D}$ is τ-finite dimensional. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all integrable in the nonunital case. Strictly speaking, this definition describes *local* (k,∞)-summability, however we use the terminology (k,∞)-summable to be consistent with the unital case.

We need to briefly discuss the Dixmier trace, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [CPS2]. For $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, the function

$$F_T : t \mapsto \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds$$

is bounded. For certain generalised limits $\omega \in L^\infty(\mathbf{R}_*^+)^*$, we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is the Dixmier trace associated to the semifinite normal trace τ, denoted τ_ω, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace τ_ω is defined on the ideal $\mathcal{L}^{(1,\infty)}$, and vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by $\fint$. So if $T \in \mathcal{L}^{(1,\infty)}$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbf{R}_*^+)^*$ we have

$$\tau_\omega(T) = \omega(F_T) = \fint T.$$

Example The Dirac operator on the k-torus. Let γ^j, $j = 1, \ldots, k$, be generators of the Clifford algebra of $\mathbf{R}^k$ with the usual Euclidean inner product. Form the Dirac operator on spinors $\mathcal{D} = \sum_{j=1}^{k} \gamma^j \frac{\partial}{\partial \theta^j}$, which acts on $L^2(\mathbf{T}^k) \otimes \mathbf{C}^{2^{[k/2]}}$, and for $n \in \mathbf{Z}^k$, let $n^2 \in \mathbf{N}$ denote the sum $n^2 = \sum_{i=1}^{k} n_i^2$ of the squares of the coordinates of n. Then it is well known that the spectrum of $\mathcal{D}^2$ consists of eigenvalues $\{n^2 \in \mathbf{N}\}$, where each $n \in \mathbf{Z}^k$ is counted once. A careful calculation taking account of the multiplicities, [La], shows that using the standard operator trace, the function $F_{(1+\mathcal{D}^2)^{-k/2}}$ is

$$\frac{1}{\log(|\{n : |n| \le N\}|)} \sum_{|n|=0}^{N} (1 + n^2)^{-k/2} = \frac{2^{[k/2]} vol(S^{k-1})}{k \log N} \sum_{m=0}^{N} (1 + m^2)^{-1/2} + o(1)$$

and this is bounded. Hence $(1 + \mathcal{D}^2)^{-k/2} \in \mathcal{L}^{(1,\infty)}$ and

$$\text{Trace}_\omega((1 + \mathcal{D}^2)^{-k/2}) = \int (1 + \mathcal{D}^2)^{-k/2} = \frac{2^{[k/2]} vol(S^{k-1})}{k} = \frac{2^{[k/2]} vol(S^{k-1})}{(2\pi)^k k} vol(\mathbf{T}^k).$$

Numerous properties of local algebras are established in [R1, R2]. The introduction of quasi-local algebras in [GGISV] led to a review of the validity of many of these results for quasi-local algebras. Most of the summability results of [R2] are valid in the quasi-local setting. In addition, the summability results of [R2] are also valid for general semifinite spectral triples since they rely only on properties of the ideals $\mathcal{L}^{(p,\infty)}$, $p \ge 1$, [C, CPS2], and the trace property. We quote the version of the summability results from [R2] that we require below.

Proposition 4.10 ([R2]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞, local (k, ∞)-summable semifinite spectral triple. Let $T \in \mathcal{N}$ satisfy $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. Then*

$$T(1 + \mathcal{D}^2)^{-k/2} \in \mathcal{L}^{(1,\infty)}.$$

For $Re(s) > k$, $T(1 + \mathcal{D}^2)^{-s/2}$ is trace class. If the limit

$$(4) \qquad \lim_{s \to k/2^+} (s - k/2)\tau(T(1 + \mathcal{D}^2)^{-s})$$

exists, then it is equal to

$$\frac{k}{2} \int T(1 + \mathcal{D}^2)^{-k/2}.$$

In addition, for any Dixmier trace τ_ω, the function

$$a \mapsto \tau_\omega(a(1 + \mathcal{D}^2)^{-k/2})$$

defines a trace on $\mathcal{A}_c \subset \mathcal{A}$.

5. CONSTRUCTING A C^*-MODULE AND A KASPAROV MODULE

Let $A = C^*(\Lambda)$ where Λ is a locally finite locally convex k-graph. Let $F = C^*(\Lambda)^\gamma$ be the fixed point subalgebra for the gauge action. Finally, let $A_c = \text{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$ and let $F_c = \text{span}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\} = F \cap A_c$ so that A and F are the C^*-completions of A_c and F_c. Note that the expectation $\Phi : A \to F$ outlined at the end of Section 2 restricts to an expectation, also denoted Φ of A_c onto F_c.

For $q \in \mathbf{Q}$, we denote by $[q]$ the integer part $\max\{n \in \mathbf{Z} : n \le q\}$ of q. We make $A^{2^{[k/2]}} = \mathbf{C}^{2^{[k/2]}} \otimes A$ a right inner product-F-module. The right action of F on A is by right multiplication. The inner product is defined by

$$(x|y)_R := \sum_{j=1}^{2^{[k/2]}} \Phi(x_j^* y_j) \in F.$$

It is simple to check the requirements that $(\cdot|\cdot)_R$ defines an F-valued inner product on $A^{2^{[k/2]}}$. The requirement $(x|x)_R = 0 \Rightarrow x = 0$ follows from the faithfulness of Φ.

Definition 5.1. *Define X to be the completion of $A^{2^{[k/2]}}$ to a C^*-module over F for the C^*-module norm*

$$\|x\|_X^2 := \|(x|x)_R\|_A = \|(x|x)_R\|_F = \|\sum_{i=1}^{2^{[k/2]}} \Phi(x_i^* x_i)\|_F.$$

Define X_c to be the pre-C^*-module over F_c with linear space $A_c^{2^{[k/2]}}$ and the inner product $(\cdot|\cdot)_R$.

Remark Typically, the action of F does not map X_c to itself, so we may only consider X_c as an F_c module. This is a reflection of the fact that F_c and A_c are quasilocal not local.

Remark Frequently we will define an operator T on the F module A, and implicitly extend T to X by $\mathrm{id}_{2^{[k/2]}} \otimes T$, where $\mathrm{id}_{2^{[k/2]}}$ is the identity operator in the matrix algebra $M_{2^{[k/2]}}(\mathbf{C})$.

Remark There is an irreducible representation γ of the complex Clifford algebra $\mathbf{C}\mathit{liff}_k = \mathit{Cliff}(\mathbf{C}^k)$ on $\mathbf{C}^{2^{[k/2]}}$, and tensoring this representation by the identity map on A, this extends to a representation on X as adjointable operators. We employ the convention that

$$\gamma^l \gamma^j + \gamma^j \gamma^l := \gamma(e^l)\gamma(e^j) + \gamma(e^j)\gamma(e^l) = -2\delta^{lj} \, \mathrm{id}_{2^{[k/2]}} \, .$$

When k is even the operator $\omega_\mathbf{C} := i^{[(k+1)/2]} \gamma^1 \cdots \gamma^k$ is self-adjoint, has $\omega_\mathbf{C}^2 = \mathrm{id}_{2^{[k/2]}}$ and $\gamma^j \omega_\mathbf{C} = -\omega_\mathbf{C} \gamma^j$ for $j = 1, \ldots, k$. When k is odd, $\omega_\mathbf{C}$ is central in the Clifford algebra, and we choose the representation with $\omega_\mathbf{C} = 1$.

The map $a \mapsto 1_{2^{[k/2]}} \otimes a$ is an isometric inclusion of A into $\mathbf{C}^{2^{[k/2]}} \otimes A = A^{2^{[k/2]}}$, which in turn is dense in X by definition. The inclusion $\iota : A \to X$ is continuous since

$$\|a\|_X^2 = \|\Phi(a^* a)\|_F \le \|a^* a\|_A = \|a\|_A^2.$$

We can also define the gauge action γ on $A \subset X$, and as

$$\begin{aligned}
\|\gamma_z(a)\|_X^2 &= \|\Phi((\gamma_z(a))^*(\gamma_z(a)))\|_F = \|\Phi(\gamma_z(a^*)\gamma_z(a))\|_F \\
&= \|\Phi(\gamma_z(a^* a))\|_F = \|\Phi(a^* a)\|_F = \|a\|_X^2,
\end{aligned}$$

for each $z \in \mathbf{T}^k$, the action of γ_z is isometric on $A \subset X$ and so extends to a unitary U_z on X. This unitary is F-linear and adjointable, and we obtain a strongly continuous action of $\mathbf{T}^k$ on X, which we still denote by γ.

For each $n \in \mathbf{Z}^k$, define an operator Φ_n on X by

$$\Phi_n(x) = \frac{1}{(2\pi)^k} \int_{\mathbf{T}^k} z^{-n} \gamma_z(x) d^k \theta, \quad z_j = e^{i\theta_j}, \quad x \in X.$$

Observe that on generators we have

(5)
$$\Phi_n(S_\alpha S_\beta^*) = \begin{cases} S_\alpha S_\beta^* & d(\alpha) - d(\beta) = n \\ 0 & d(\alpha) - d(\beta) \neq n \end{cases}.$$

Remark If (Λ, d) is a finite k-graph with no cycles, then for n sufficiently large there are no paths of degree n and so $\Phi_n = 0$. This will obviously simplify many of the convergence issues below.

The proof of the following Lemma is identical to that of [PRen, Lemma 4.2].

Lemma 5.2. *The operators Φ_n are adjointable endomorphisms of the F-module X such that $\Phi_n^* = \Phi_n = \Phi_n^2$ and $\Phi_n \Phi_m = \delta_{n,m} \Phi_n$. For each subset $K \subset \mathbf{Z}^k$, the sum $\sum_{n \in K} \Phi_n$ converges strictly to a projection Φ_K in the endomorphism algebra. Moreover, the projection $\Phi_{\mathbf{Z}^k}$ corresponding to $K = \mathbf{Z}^k$ is the identity operator on X.*

Corollary 5.3. *Let $x \in X$. Then with $x_n = \Phi_n x$ the sum $\sum_{n \in \mathbf{Z}^k} x_n$ converges in X to x.*

5.1. **The Kasparov Module.** As we did in Section 4, for $n \in \mathbf{Z}^k$, we write $n^2 = \sum_{j=1}^k n_j^2$ and $|n| = \sqrt{n^2}$.

The theory of unbounded operators on C^*-modules that we require is all contained in Lance's book, [L, Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 5.4. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D}$: dom $\mathcal{D} \subset Y \to Y$ is a B-linear operator defined on a dense B-submodule dom $\mathcal{D} \subset Y$. The operator $\mathcal{D}$ is closed if the graph*

$$G(\mathcal{D}) = \{(x, \mathcal{D}x)_R : x \in \text{dom } \mathcal{D}\}$$

is a closed submodule of $Y \oplus Y$.

Given a densely defined unbounded operator $\mathcal{D}$: dom $\mathcal{D} \subset Y \to Y$, define a submodule

$$\text{dom } \mathcal{D}^* := \{y \in Y : \exists z \in Y \text{ such that } \forall x \in \text{dom } \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \text{dom } \mathcal{D}^*$ define $\mathcal{D}^* y = z$. Given $y \in \text{dom } \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^*$ is well-defined, and moreover is closed.

Definition 5.5. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D}$: dom $\mathcal{D} \subset Y \to Y$ is symmetric if for all $x, y \in \text{dom } \mathcal{D}$*

$$(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R.$$

A symmetric operator $\mathcal{D}$ is self-adjoint if $\text{dom } \mathcal{D} = \text{dom } \mathcal{D}^*$ (and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^* \mathcal{D})$ has dense range.

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatically satisfied, [L, Chapter 9].

With these definitions in hand, we return to our C^*-module X. The proof of the following Proposition is an exact analogue of [PRen, Proposition 4.6].

Proposition 5.6. *Let X be the right C^*-F-module of Definition 5.1. Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \{x = \sum_{n \in \mathbf{Z}^k} x_n : \| \sum_{n \in \mathbf{Z}^k} n^2 (x_n | x_n)_R \| < \infty \}.$$

For $x \in X_{\mathcal{D}}$ define

$$\mathcal{D}x = \sum_{n \in \mathbf{Z}^k} \gamma(in) x_n = i \sum_{n \in \mathbf{Z}^k} \sum_{j=1}^{k} \gamma^j n_j x_n.$$

Then $\mathcal{D} : X_{\mathcal{D}} \to X$ is self-adjoint and regular.

Remark For $n \in \mathbf{Z}^k$, the restriction of the map $\mathcal{D}$ to $\Phi_n X$ implements Clifford multiplication by the vector $in \in \mathbf{C}^k$. Any $S_\alpha S_\beta^* \in A_c$ is in $X_{\mathcal{D}}$ and

$$\mathcal{D} S_\alpha S_\beta^* = i \sum_{j=1}^{k} \gamma^j (d(\alpha)_j - d(\beta)_j) S_\alpha S_\beta^*$$

as the reader will easily verify. Thus we have

$$\mathcal{D}^2 \Phi_n x = \sum_{j=1}^{k} n_j^2 \Phi_n x = n^2 \Phi_n x.$$

There is a continuous functional calculus for self-adjoint regular operators, [L, Theorem 10.9], and we use this to obtain spectral projections for $\mathcal{D}^2$ at the C^*-module level. Let $f_m \in C_c(\mathbf{R})$ be 1 in a small neighbourhood of $m \in \mathbf{Z}$ and zero on $(-\infty, m-1/2] \cup [m+1/2, \infty)$. Then it is clear that

$$\sum_{n \in \mathbf{Z}^k, \ n^2 = m} \Phi_n = f_m(\mathcal{D}^2).$$

The next Lemma is the first place where we need our k-graph to be locally finite and have no sinks. It is also the point where the generalisation from the graph case differs the most.

Lemma 5.7. *Assume that the k-graph (Λ, d) is locally finite, locally convex and has no sinks. For all $a \in A$ and $n \in \mathbf{Z}^k$, $a\Phi_n \in \mathrm{End}_F^0(X)$, the compact endomorphisms of the right F-module X. If $a \in A_c$ then $a\Phi_n$ is finite rank.*

Remarks 5.8. If we were employing the A-valued inner product on X, then each $a \in A$ would be compact, and Lemma 5.7 would be an immediate corollary of the fact that the compacts form an ideal. However, with our choice of inner product, with values in F, *no $a \in A$ acts as a compact endomorphism*, except in some extreme examples.

Proof. Let $n \in \mathbf{Z}^k$ and write $n = n_1 + n_2$ with $n_1 \geq 0$ and $n_2 < 0$. We will see that the precise choice of n_1, n_2 is largely irrelevant. For $v \in \Lambda^0$, let $|v|_n$ denote the number of paths $\rho \in \Lambda$ with $d(\rho) = n$ and $s(\rho) = v$, i.e. $|v|_n = |\Lambda^n v|$. Since Λ has no sinks and is locally finite, for all n and all v we have $0 < |v|_n < \infty$.

Now define, for $n = n_1 + n_2$ as above,

$$T_{v,n_1,n_2} = \sum_{d(\alpha)=n_1, d(\beta)=-n_2, s(\alpha)=s(\beta), r(\alpha)=v} \frac{1}{|s(\beta)|_{-n_2}} \Theta^R_{S_\alpha S_\beta^*, S_\alpha S_\beta^*},$$

where for $x, y, z \in X$

$$\Theta^R_{x,y} z := x(y|z)_R,$$

defines a rank one operator. Observe that since Λ is locally finite this is a finite sum of rank one operators and so finite rank. We claim that $T_{v,n_1,n_2} = p_v \Phi_n$. It suffices to prove that the difference $p_v \Phi_n - T_{v,n_1,n_2}$ vanishes on $X_c \subset X$. That is, we just need to show that $(p_v \Phi_n - T_{v,n_1,n_2}) S_\mu S_\nu^* = 0$ for all μ, ν. So first we compute, with $q = d(\alpha) \vee d(\mu)$,

$$S_\alpha^* S_\mu = \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\sigma S_\rho^*$$

by [RSY, Proposition 3.5 and Remarks 3.8(2)]. Next consider

$$\Phi(S_\beta S_\alpha^* S_\mu S_\nu^*) = \Phi(S_\beta \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\sigma S_\rho^* S_\nu^*).$$

This is zero unless $d(\beta) + d(\sigma) - d(\rho) - d(\nu) = 0$. Now $d(\sigma) - d(\rho) = d(\mu) - d(\alpha)$ so

$$\Phi(S_\beta S_\alpha^* S_\mu S_\nu^*) = \delta_{d(\mu)-d(\nu), d(\alpha)-d(\beta)} S_\beta \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\sigma S_\rho^* S_\nu^*.$$

Of course, $d(\alpha) - d(\beta) = n$. Since each $S_\beta^* S_\beta = p_{s(\beta)}$, we can perform the sum over β:

$$\sum_{\alpha,\beta} \frac{1}{|s(\beta)|_{-n_2}} p_v \Theta_{S_\alpha S_\beta^*, S_\alpha S_\beta^*} S_\mu S_\nu^* = \sum_{\alpha,\beta} \frac{1}{|s(\beta)|_{-n_2}} \delta_{d(\mu)-d(\nu),n} p_v S_\alpha S_\beta^* S_\beta \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\sigma S_\rho^* S_\nu^*$$

$$= \sum_{\alpha} \delta_{d(\mu)-d(\nu),n} p_v S_\alpha \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\sigma S_\rho^* S_\nu^*$$

$$= \sum_{\alpha} \delta_{d(\mu)-d(\nu),n} p_v \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\mu S_\rho S_\rho^* S_\nu^*.$$

If we suppose that a given α has no common extensions with μ, then this particular term in the sum contributes zero. Summing over all α (of fixed length n_1) with common extensions with μ yields

$$\sum_{\alpha} \delta_{d(\mu)-d(\nu),n} p_v \sum_{\alpha\sigma=\mu\rho, \alpha\sigma\in\Lambda^q} S_\mu S_\rho S_\rho^* S_\nu^* = p_v \sum_{\rho\in\Lambda^{q-d(\mu)}, r(\rho)=s(\mu)} S_\mu S_\rho S_\rho^* S_\nu^* = S_\mu S_\nu^*.$$

Hence we conclude that

$$\sum_{\alpha,\beta} \frac{1}{|s(\beta)|_{-n_2}} p_v \Theta_{S_\alpha S_\beta^*, S_\alpha S_\beta^*} S_\mu S_\nu^* = \delta_{d(\mu)-d(\nu),n} p_v S_\mu S_\nu^*$$

$$= p_v \Phi_n S_\mu S_\nu^*.$$

As μ, ν were arbitrary paths, this shows that $p_v \Phi_n$ is a finite rank endomorphism. For arbitrary $a = \sum c_j S_{\mu_j} S_{\nu_j}^*$, where the sum is finite, we may apply the same reasoning to each $p_{s(\nu_j)}$ to see that $a\Phi_n$ is finite rank for all $a \in A_c$.

To see that $a\Phi_k$ is compact for all $a \in A$, recall that every $a \in A$ is a norm limit of a sequence $\{a_i\}_{i\geq 0} \subset A_c$. Thus for any $n \in \mathbf{Z}^k$ $a\Phi_n = \lim_{i\to\infty} a_i\Phi_n$ and so is compact. $\qquad \square$

Lemma 5.9. *Assume that the k-graph (Λ, d) is locally finite and has no sinks. For all $a \in A$, $a(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F-module X.*

Proof. First let $a = p_v$ for $v \in \Lambda^0$. Then the sum

$$R_{v,N} := p_v \sum_{|n|=0}^{N} \Phi_n (1 + n^2)^{-1/2}$$

is finite rank, by Lemma 5.7. We will show that the sequence $\{R_{v,N}\}_{N\geq 0}$ is convergent with respect to the operator norm $\| \cdot \|_{End}$ of endomorphisms of X. Indeed, assuming that $M > N$,

$$\|R_{v,N} - R_{v,M}\|_{End} = \|p_v \sum_{|n|=N+1}^{M} \Phi_n (1 + n^2)^{-1/2}\|_{End}$$

$$(6) \qquad\qquad\qquad \leq (1 + (N+1)^2)^{-1/2} \to 0,$$

since the ranges of the $p_v \Phi_n$ are orthogonal for different n. Thus, using the argument from Lemma 5.7, $a(1 + \mathcal{D}^2)^{-1/2} \in End_F^0(X)$ for all $a \in A_c$. Letting $\{a_i\}$ be a Cauchy sequence from A_c, we have

$$\|a_i(1 + \mathcal{D}^2)^{-1/2} - a_j(1 + \mathcal{D}^2)^{-1/2}\|_{End} \leq \|a_i - a_j\|_{End} = \|a_i - a_j\|_A \to 0,$$

since $\|(1 + \mathcal{D}^2)^{-1/2}\| \leq 1$. Thus the sequence $a_i(1 + \mathcal{D}^2)^{-1/2}$ is Cauchy in norm and we see that $a(1 + \mathcal{D}^2)^{-1/2}$ is compact for all $a \in A$. $\qquad \square$

Proposition 5.10. *Assume that the k-graph (Λ, d) is locally finite and has no sinks. Let $V = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. Then (X, V) defines a class in $KK^{k \bmod 2}(A, F)$.*

Proof. We refer to [K] for more information. We need to show that various operators belong to $End_F^0(X)$. First, $V - V^* = 0$, so $a(V - V^*)$ is compact for all $a \in A$. Also $a(1 - V^2) = a(1 + \mathcal{D}^2)^{-1}$ which is compact from Lemma 5.9 and the boundedness of $(1 + \mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V, a]$ is compact for all $a \in A$. First we suppose that $a \in A_c$. Then we have

$$[V, a] = [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2} - \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2}$$

$$= b_1(1 + \mathcal{D}^2)^{-1/2} + V b_2(1 + \mathcal{D}^2)^{-1/2},$$

where $b_1 = [\mathcal{D}, a] \in A_c$ and $b_2 = [(1 + \mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 5.9 will show that $[V, a]$ is compact for all $a \in A_c$. So consider the action of $[(1 + \mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1 + \mathcal{D}^2)^{-1/2}$ on $x = \sum_{n \in \mathbf{Z}^k} x_n$. We find

$$\sum_{n \in \mathbf{Z}^k} [(1 + \mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1 + \mathcal{D}^2)^{-1/2} x_n$$

$$= \sum_{n \in \mathbf{Z}^k} \left((1 + (d(\mu) - d(\nu) + n)^2)^{1/2} - (1 + n^2)^{1/2}\right)(1 + n^2)^{-1/2} S_\mu S_\nu^* x_n$$

(7)
$$= \sum_{n \in \mathbf{Z}^k} f_{\mu,\nu}(n) S_\mu S_\nu^* \Phi_n x.$$

The function

$$f_{\mu,\nu}(n) = \left((1 + (d(\mu) - d(\nu) + n)^2)^{1/2} - (1 + n^2)^{1/2}\right)(1 + n^2)^{-1/2}$$

goes to zero as $n^2 \to \infty$. As the $S_\mu S_\nu^* \Phi_n$ are finite rank with orthogonal ranges (for different n), the sum in (7) converges in the endomorphism norm, and so converges to a compact endomorphism. For general $a \in A_c$ we write a as a finite linear combination of generators $S_\mu S_\nu^*$, and apply the above reasoning to each term in the sum to find that $[(1 + \mathcal{D}^2)^{1/2}, a]$ is a compact endomorphism for all $a \in A_c$.

Now let $a \in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i \geq 0} \subset A_c$. Then

$$\|[V, a_i - a_j]\|_{End} \leq 2\|a_i - a_j\|_{End} \to 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact.

It is also clear from the construction that if k is even, the Kasparov module is even (with grading given by $\omega_{\mathbf{C}}$) and so belongs to $KK^0(A, F)$, while when k is odd, the Kasparov module belongs to $KK^1(A, F)$. $\qquad\square$

6. The Gauge Spectral Triple of a k-Graph Algebra

In this section we will construct a semifinite spectral triple for those locally convex k-graph C^*-algebras which possess a faithful, semifinite, lower-semicontinuous, gauge invariant trace, τ. Recall from Proposition 3.8 that such traces arise from faithful k-graph traces.

We will begin with the right F_c module X_c. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that (Λ, d) is locally finite and has no sinks. This ensures, by Lemma 5.7 that for all $a \in A$ and $n \in \mathbf{Z}^k$ the endomorphisms $a\Phi_n$ of X are compact endomorphisms.

We define a $\mathbf{C}$-valued inner product on X_c by

$$\langle x, y \rangle := \tau((x|y)_R) = \sum_{j=1}^{2^{[k/2]}} \tau(\Phi(x_j^* y_j)) = \sum_{j=1}^{2^{[k/2]}} \tau(x_j^* y_j).$$

Observe that this inner product is linear in the second variable. We define the Hilbert space $\mathcal{H} = L^2(X, \tau)$ to be the completion of X_c in the norm coming from the inner product.

Lemma 6.1. *The C^*-algebra $A = C^*(\Lambda)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of A. Moreover, any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on $\mathcal{H}$.*

Proof. The first statement follows from the proof of Proposition 3.8. Now let T be an endomorphism of X leaving X_c invariant. Then [RW, Cor 2.22],

$$(Tx|Ty)_R \leq \|T\|_{End}^2 (x|y)_R$$

in the algebra F. Now the norm of T as an operator on $\mathcal{H}$, denoted $\|T\|_\infty$, can be computed in terms of the endomorphism norm of T by

$$\|T\|_\infty^2 := \sup_{\|x\|_{\mathcal{H}} \leq 1} \langle Tx, Tx \rangle = \sup_{\|x\|_{\mathcal{H}} \leq 1} \tau((Tx|Tx)_R)$$

(8)
$$\leq \sup_{\|x\|_{\mathcal{H}} \leq 1} \|T\|_{End}^2 \tau((x|x)_R) = \|T\|_{End}^2.$$

$\square$

Corollary 6.2. *The endomorphisms $\{\Phi_n\}_{n \in \mathbf{Z}^k}$ define mutually orthogonal projections on $\mathcal{H}$. For any $K \subset \mathbf{Z}^k$ the sum $\sum_{n \in K} \Phi_n$ converges strongly to a projection Φ_K in $\mathcal{B}(\mathcal{H})$. The projection $\Phi_{\mathbf{Z}^k}$ corresponding to $K = \mathbf{Z}^k$ is equal to $\mathrm{id}_{\mathcal{H}}$, so that for all $x \in \mathcal{H}$ the sum $\sum_n \Phi_n x$ converges in norm to x.*

Proof. As in Lemma 5.2, we can use the continuity of the Φ_n on $\mathcal{H}$, which follows from Lemma 6.1, to see that the relation $\Phi_n \Phi_m = \delta_{n,m} \Phi_n$ extends from $X_c \subset \mathcal{H}$ to $\mathcal{H}$. The proof of the strong convergence of sums of Φ_n's is just as in Lemma 5.2 after replacing the C^*-module norm with the Hilbert space norm. $\square$

Lemma 6.3. *The operator $\mathcal{D}$ extends to a closed unbounded self-adjoint operator on $\mathcal{H}$. The closure of the operator $\mathcal{D}|_{X_c}$ is $\mathcal{D}$.*

Proof. The proof is essentially the same as the C^*-module version, Lemma 5.6. By replacing the C^*-module norm and the C^*-Cauchy-Schwartz inequality with the Hilbert space analogues, the proof that $\mathcal{D}$ is closed goes through as before. We then define dom $\mathcal{D}$ to be the completion of X_c in the norm

$$x \to \|x\|_{\mathcal{H}, \mathcal{D}} := \|x\|_{\mathcal{H}} + \|\mathcal{D}x\|_{\mathcal{H}}.$$

The proofs of symmetry and self-adjointness now follow just as in the C^*-module case. The last statement follows from the definition of dom $\mathcal{D}$. $\square$

The Hilbert space $\mathcal{H}$ and operator $\mathcal{D}$ are two of the ingredients of our spectral triple. We also need a $*$-algebra. In fact A_c will do the job, but it also has a natural completion $\mathcal{A}$ which is useful too. To prove both these assertions we need the following lemma. The proof is the same as [PRen, Lemma 5.4].

Lemma 6.4. *Let $\mathcal{H}, \mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well-defined on X_c, and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_{\mathcal{H}} \leq \left| d(\alpha) - d(\beta) \right|.$$

Similarly, $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_{\mathcal{H}} = \left| d(\alpha) - (\beta) \right|$.

Corollary 6.5. *The algebra A_c is contained in the smooth domain of the derivation δ where for $T \in \mathcal{B}(\mathcal{H})$, $\delta(T) = [|\mathcal{D}|, T]$. That is*

$$A_c \subseteq \bigcap_{n \geq 0} \operatorname{dom} \delta^n.$$

Definition 6.6. *Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ-topology. By Lemma 4.4, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.*

Lemma 6.7. *If $a \in \mathcal{A}$ then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geq 0$. If $\phi \in \mathcal{A}$ satisfies $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is $A \otimes M_{2^{[k/2]}}(\mathbf{C})$. In particular, $\mathcal{A}$ is quasi-local.*

We leave the straightforward proofs of these statements to the reader.

At this point we have most of the structure required to define a semifinite local spectral triple. The one remaining piece of information we require is the compactness of $a(\lambda - \mathcal{D})^{-1}$, $\lambda \in \mathbf{C} \setminus \mathbf{R}$, $a \in \mathcal{A}$, relative to some trace on some von Neumann algebra to which $\mathcal{D}$ is affiliated. There is a canonical choice of von Neumann algebra and trace, and for this choice $a(1 + \mathcal{D}^2)^{-k/2}$ is in the domain of the Dixmier trace for all $a \in \mathcal{A}$.

6.1. **Traces and Compactness Criteria.** We continue to assume that (Λ, d) is a locally convex locally finite k-graph with no sinks and that τ is a faithful, semifinite, lower-semicontinuous, gauge invariant trace on $C^*(\Lambda)$. We will define a von Neumann algebra $\mathcal{N}$ with a faithful semifinite normal trace $\tilde{\tau}$ so that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are as defined in the last subsection. Moreover the operator $\mathcal{D}$ will be affiliated to $\mathcal{N}$. To state the theorem, we need some preliminary definitions and results.

Definition 6.8. *Let $End_F^{00}(X_c)$ denote the algebra of finite rank operators on X_c acting on $\mathcal{H}$. Define $\mathcal{N} = (End_F^{00}(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.*

Definition 6.9. *Let $T \in \mathcal{N}$. For $n \in \mathbf{N}^k$ and $v \in \Lambda^0$, let $|v|_n$ denote the number of paths of degree n with source v. Let $\Lambda \times_s^{\min} \Lambda$ denote the set of pairs*

$$\{(\alpha, \beta) \in \Lambda : s(\alpha) = s(\beta), d(\alpha) \wedge d(\beta) = 0\}.$$

For $(\alpha, \beta) \in \Lambda \times_s^{\min} \Lambda$, define

$$\omega_{\alpha,\beta}(T) = \frac{1}{|s(\alpha)|_{d(\beta)}} \langle s_\alpha s_\beta^*, T s_\alpha s_\beta^* \rangle.$$

Note that if $d(\alpha) = d(\beta) = 0$, then $\alpha = \beta = v$ for some $v \in \Lambda^0$, and since $s_v = p_v$ by convention, we have $\omega_{v,v}(T) = \langle p_v, Tp_v \rangle$. Define

$$(9) \qquad \tilde{\tau} : \mathcal{N}_+ \to [0, \infty], \quad by \quad \tilde{\tau}(T) = \lim_{L \nearrow \Lambda \times_s^{\min} \Lambda} \sum_{(\alpha,\beta) \in L} \omega_{\alpha,\beta}(T)$$

where L increases over the net of finite subsets of $\Lambda \times_s^{\min} \Lambda$.

Remarks

(1) For $T, S \in \mathcal{N}_+$ and $\lambda \geq 0$ we have

$$\tilde{\tau}(T + S) = \tilde{\tau}(T) + \tilde{\tau}(S) \quad \text{and} \quad \tilde{\tau}(\lambda T) = \lambda \tilde{\tau}(T) \quad \text{where} \quad 0 \times \infty = 0.$$

(2) Note that for $\mu \in \Lambda$, we have $\omega_{\mu,s(\mu)}(T) = \langle s_\mu, Ts_\mu \rangle$ and $\omega_{s(\mu),\mu}(T) = \frac{1}{|s(\mu)|_{d(\mu)}} \langle s_\mu^*, Ts_\mu^* \rangle$. Consequently, if Λ is a 1-graph then for $\mu \in \Lambda \backslash \Lambda^0$, the map ω_μ of [PRen, Definition 5.10] is precisely $\omega_{\mu,s(\mu)} + \omega_{s(\mu),\mu}$, while for $v \in \Lambda^0$, $\omega_v = \omega_{v,v}$. In particular, for a 1-graph, (9) is just a slightly more efficient expression for the definition of $\tilde{\tau}$ of [PRen, Definition 5.10].

Theorem 6.10. *Let (Λ, d) be a locally convex locally finite k-graph with no sinks, and let τ be a faithful semifinite trace on $C^*(\Lambda)$. Let $\mathcal{N}$ be as in Definition 6.8 and let $\tilde{\tau} : \mathcal{N}^+ \to [0, \infty]$ be as in Definition 6.9. Then*

(1) *$\tilde{\tau}$ defines a faithful normal semifinite trace on $\mathcal{N}$.*
(2) *$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ (k, ∞)-summable odd local semifinite spectral triple relative to $(\mathcal{N}, \tilde{\tau})$.*
(3) *For all $a \in \mathcal{A}$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is not trace class.*

Suppose that $v \in \Lambda^0$ satisfies $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$. Then

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}) = C_k\tau(p_v),$$

where $\tilde{\tau}_\omega$ is any Dixmier trace associated to $\tilde{\tau}$, and

$$C_k = \frac{2^{[k/2]}vol(S^{k-1})}{k}$$

Remark The hypothesis that $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$ is perhaps somewhat opaque. This theorem generalises [PRen, Theorem 5.8], which requires that the vertex v has "no sinks downstream" to ensure that $p_v = \sum_{s(\alpha)=v, |\alpha|=n} s_\alpha s_\alpha^*$ for all $n \in \mathbf{N}$. The hypothesis that $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$ has precisely the same effect (consider relation (CK4)). Indeed this is precisely the notion that the $\Lambda^{\leq n}$ notation was developed to capture: $\Lambda^{\leq n}$ is supposed to consist of all paths of degree n together with all paths whose degree is less than n because they originate at a source in direction n [RSY].

Proposition 6.11. *The function $\tilde{\tau} : \mathcal{N}_+ \to [0, \infty]$ defines a faithful normal semifinite trace on $\mathcal{N}$. Moreover,*

$$End_F^{00}(X_c) \subset \mathcal{N}_{\tilde{\tau}} := \text{span}\{T \in \mathcal{N}_+ : \tilde{\tau}(T) < \infty\},$$

the domain of definition of $\tilde{\tau}$, and

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x \rangle = \tau(y^*x), \quad x, y \in X_c.$$

The proof of this important, but technical, result is extremely similar to that of [PRen, Proposition 5.11], differing only in the details of the calculations establishing the analogue of [PRen, Equation (18)] and showing that $\tilde{\tau}(\Theta^R_{x,y}) = \tau(y^*x)$ for all x, y.

Lemma 6.12. *Let (Λ, d) be a locally convex locally finite k-graph with no sinks and a faithful gauge invariant trace τ on $C^*(\Lambda)$. Let $v \in \Lambda^0$ and $n \in \mathbf{Z}^k$. Then*

$$\tilde{\tau}(p_v\Phi_n) \leq \tau(p_v)$$

with equality when $v\Lambda^{\leq p} = v\Lambda^p$.

Proof. Let $n = n_+ + n_-$ where $n_+ \geq 0$, $n_- \leq 0$, and $n_+ \vee -n_- = n_+ - n_-$. By Lemma 5.7 and Proposition 6.11 we have

$$
\begin{aligned}
\tilde{\tau}(p_v\Phi_n) &= \tilde{\tau}\left(p_v \sum_{d(\alpha)=n_+, d(\beta)=-n_-} \frac{1}{|s(\beta)|_{-n_-}} \Theta_{S_\alpha S_\beta^*, S_\alpha S_\beta^*}\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, d(\beta)=-n_-} \frac{1}{|s(\beta)|_{-n_-}} (S_\alpha S_\beta^* | p_v S_\alpha S_\beta^*)_R\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, d(\beta)=-n_-} \frac{1}{|s(\beta)|_{-n_-}} \Phi(S_\beta S_\alpha^* p_v S_\alpha S_\beta^*)\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, d(\beta)=-n_-, r(\alpha)=v} \frac{1}{|s(\beta)|_{-n_-}} S_\alpha S_\beta^* S_\beta S_\alpha^* p_v\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, r(\alpha)=v} S_\alpha S_\alpha^* p_v\right).
\end{aligned}
$$

If there are no sources within $|n_+|$ of v, then $\sum_{d(\alpha)=n_+, r(\alpha)=v} S_\alpha S_\alpha^* = p_{r(\alpha)} = p_v$. Otherwise the sum on the right is strictly less than p_v. So

$$\tilde{\tau}(p_v\Phi_n) \leq \tau(p_v)$$

with equality when there are no sources within $|n_+|$ of v. $\square$

Proposition 6.13. *Assume that the locally convex k-graph (Λ, d) is locally finite, has no sinks and has a faithful gauge invariant trace on $C^*(\Lambda)$. For all $a \in \mathcal{A}_c$ the operator $a(1 + \mathcal{D}^2)^{-k/2}$ is in the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau})$. When $v \in \Lambda^0$ satisfies $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$, we have*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}) = \frac{2^{[k/2]}vol(S^{k-1})}{k}\tau(p_v).$$

Proof. It suffices to show this for a projection $a = p_v$ for $v \in \Lambda^0$, and extending to more general $a \in A_c$ using the arguments of Lemma 5.7. We compute the partial sums defining the trace of $p_v(1 + \mathcal{D}^2)^{-k/2}$. Lemma 6.12 gives us

$$(10) \qquad \tilde{\tau}\left(p_v \sum_{|n| \leq N} (1 + n^2)^{-k/2} \Phi_n\right) \leq \sum_{|n| \leq N} (1 + n^2)^{-k/2} \tau(p_v).$$

We have equality when $v\Lambda^{\leq n} = v\Lambda^n$ whenever $|n| \leq N$. Since Λ has no sinks, the sequence

$$\frac{1}{\log|\{n : |n| \leq N\}|} \sum_{|n| \leq N} (1 + n^2)^{-k/2} \tilde{\tau}(p_v \Phi_k)$$

is bounded (there is at least one 'direction' in which n can increase indefinitely, so the sequence does not go to zero). Hence $p_v(1 + \mathcal{D}^2)^{-k/2} \in \mathcal{L}^{(1,\infty)}$ and for any ω-limit we have

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}) \leq \omega\text{-}\lim \frac{2^{[k/2]} vol(S^{k-1})}{k \log m} \sum_{m=0}^{N} (1 + m^2)^{-1/2} \tilde{\tau}(p_v \Phi_k).$$

When there are no sources in Λ, we have equality in Equation (10) for any $v \in \Lambda^0$ and so

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}) = \frac{2^{[k/2]} vol(S^{k-1})}{k} \tau(p_v).$$

$\square$

Computing the Dixmier trace when $v\Lambda^{\leq n}$ may be strictly larger than $v\Lambda^n$ for some n is harder.

Remark Using Proposition 4.10, one can check that

$$(11) \qquad res_{s=0} \tilde{\tau}(p_v(1 + \mathcal{D}^2)^{-k/2-s}) = \frac{k}{2} \tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}).$$

We will require this formula when we apply the local index theorem.

Corollary 6.14. *Assume Λ is locally finite, has no sources and has a faithful k-graph trace. Then for all $a \in A$, $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$.*

Proof. (of Theorem 6.10.) That we have a QC^∞ spectral triple follows from Corollary 6.5, Lemma 6.7 and Corollary 6.14. The properties of the von Neumann algebra $\mathcal{N}$ and the trace $\tilde{\tau}$ follow from Proposition 6.11. The (k,∞)-summability and the value of the Dixmier trace comes from Proposition 6.13. The locality of the spectral triple follows from Lemma 6.7. $\square$

7. THE LOCAL INDEX THEOREM FOR THE GAUGE SPECTRAL TRIPLE

The local index theorem for semifinite spectral triples described in [CPRS2, CPRS3] is relatively simple for the spectral triples constructed here. This is because of the simple way in which the triples are built using the Clifford algebra.

In the following discussion we assume we have a fixed locally finite locally convex k-graph (Λ, d) without sinks, and possessing a faithful k-graph trace. We let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the associated gauge spectral triple relative to $(\mathcal{N}, \tilde{\tau})$ constructed in the previous section.

Elementary manipulations with the Clifford variables, like those in [BCPRSW, Section 11.1], along with the Dixmier trace results, show that when k is odd we are left with only one term in the local index theorem

$$\phi_k(a_0, a_1, ..., a_k) = -\sqrt{2\pi i}\frac{1}{k!}\frac{1}{\sqrt{\pi}}\Gamma(k/2+1)res_{r=(1-k)/2}\tilde{\tau}(a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_k](1+\mathcal{D}^2)^{-(k-1)/2-r}).$$

When k is even we are left with only two terms:

$$\phi_k(a_0, a_1, ..., a_k) = \frac{1}{k!}\Gamma(k/2+1)res_{r=(1-k)/2}\tilde{\tau}(\gamma a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_k](1+\mathcal{D}^2)^{-(k-1)/2-r}),$$

$$\phi_0(a_0) = res_{r=(1-k)/2}\frac{1}{(r-(1-k)/2)}\tilde{\tau}(\gamma a_0(1+\mathcal{D}^2)^{-(k-1)/2-r}).$$

The zero component in the even case likewise vanishes for our examples. The reason for this is simply that we have complete symmetry between the ± 1 eigenspaces of γ, and so for $Re(r)$ large

$$\frac{1}{(r-(1-k)/2)}\tilde{\tau}(\gamma a_0(1+\mathcal{D}^2)^{-(k-1)/2-r}) = 0.$$

Hence in this particular case, the local index theorem is in fact computed using the Hochschild class (top component) of the Chern character, [CPRS1].

In Proposition A.3, we will describe a class of k-graphs which admit faithful graph traces. For full details, see Appendix A; for the time being we need only two facts established there: (1) that for such k-graph, the K-theory of $C^*(\Lambda)$ resides entirely on the set of ideals of $C^*(\Lambda)$ corresponding to ends (see Definition 2.6) of Λ; and (2) that for each end of Λ, the associated ideal is of the form $\mathcal{K} \otimes C(\mathbf{T}^l)$ for some $0 \le l \le k$. In particular (1) implies that it is only necessary to produce generators of K-theory corresponding to these ends.

The form of the Chern character given above shows that in odd dimensions we can detect only ends for which the number l in (2) above is odd, whilst in even dimensions we can only detect ends for which l is even. A simple analysis based on the Clifford algebra then shows that in fact we can only pair with ends where $l = k$; that is, ends which are k-tori, k-planes, or more generally k-cylinders.

Before producing an example of what this kind of index pairing can tell us, we discuss the relationship between the KK-index pairing with values in $K_0(F)$ and the semifinite index theorem.

Theorem 7.1. *Let Λ be a locally convex, locally finite k-graph without sinks which admits a faithful graph trace, let τ be the corresponding semifinite trace on $A = C^*(\Lambda)$, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the gauge spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$) obtained from Theorem 6.10. Let $(X, \mathcal{D})$ be the*

corresponding Kasparov module with class in $KK^k(A, F)$. Let $x \in K_k(A)$ be a K-theory class. Then

$$\tilde{\tau}_*([x \times (X, \mathcal{D})]) = Ch_{(\mathcal{A}, \mathcal{H}, \mathcal{D})}(Ch(x)).$$

Proof. Let us first consider the even case, where we have the $K_0(F)$-valued index of $p\mathcal{D}_+p$ on X, where p is a projection in A. The projections defining $\ker(p\mathcal{D}_+p)$ and $\mathrm{coker}(p\mathcal{D}_+p)$ are compact endomorphisms of the module X, and moreover map X_c to itself. This last assertion follows because $\mathcal{D}$ maps X_c to itself, and p may be chosen to lie in A_c which preserves X_c. The reason we can do this is that $K_0(A) = \lim K_0(\phi_n A \phi_n)$ where ϕ_n is any local approximate unit for A_c, [R1]. Hence the kernel and cokernel projections are actually endomorphisms preserving X_c.

Now such endomorphisms extend to act on the Hilbert space in a unique way. Since $\mathcal{H} = \overline{X_c}$ with respect to the norm coming from the inner product, we see that the Hilbert space kernel and cokernel projections are given by the extension of the C^*-module projections. So we have, by Lemma 6.12,

$$\tilde{\tau} - \mathrm{Index}(p\mathcal{D}_+p) = \tilde{\tau}(Q_{\ker(p\mathcal{D}_+p)} - Q_{\mathrm{coker}(p\mathcal{D}_+p)}) = \tilde{\tau}_*([\mathrm{Index}(p\mathcal{D}_+p)]) = \tilde{\tau}_*([p] \times [(X, \mathcal{D})]).$$

The argument for the odd pairing is now exactly the same, except that we consider the kernel and cokernel projections of PuP where P is the non-negative spectral projection of $\mathcal{D}$ and u is unitary. The upshot is that

$$\tilde{\tau} - \mathrm{Index}(PuP) = \tilde{\tau}_*([\mathrm{Index}(PuP)]) = \tilde{\tau}_*([u] \times [(X, \mathcal{D})]).$$

Now we wish to relate the $\tilde{\tau}$ index to the pairing of Chern characters. However, by [CPRS4], this is precisely the main theorems of [CPRS2, CPRS3] in the odd and even cases respectively, and so we are done. □

We will conclude with an example which indicates the kinds of information one might hope to obtain from the semifinite index theorem. In order to present the example explicitly, we first produce representatives for generators of K-theory coming from ends of graphs satisfying Proposition A.3. For this we need generators of the K-theory of ordinary tori (those for planes are of course well known). In fact we really only need those generators which pair with the Dirac class. These in turn can all be obtained, using the universal coefficient theorem and the fact that the K-theory of tori is free abelian, by using iterated products with the circle.

We illustrate this with a specific example; the nontrivial generator of $K^2(\mathbf{T}^2) = \mathbf{Z}^2$ (the other products are simpler). We wish to compute the product of $[u] \in K_1(C(\mathbf{T}^1))$ with itself to obtain the nontrivial element of $K_0(\mathbf{T}^2)$. Let $\theta, \phi \in [0, 2\pi]$, and set $z = e^{i\theta}$. Then $\theta \to z$ represents the generator $[u]$. Define

$$K(\theta) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y(\phi, \theta) = e^{i\phi K(\theta)/4} e^{iS/4}.$$

Then the product $[u] \times [u]$ is the class of the projection

$$P(\phi, \theta) = Y(\phi, \theta)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y(\phi, \theta).$$

A lengthy computation shows that

$$P(\phi, \theta) = \begin{pmatrix} 1 - \sin^2(\phi/2)\cos^2(\theta/2) & \frac{i}{2}\sin(\phi)\cos^2(\theta/2) - \frac{1}{2}\sin(\phi/2)\sin(\theta) \\ -\frac{i}{2}\sin(\phi)\cos^2(\theta/2) - \frac{1}{2}\sin(\phi/2)\sin(\theta) & \sin^2(\phi/2)\cos^2(\theta/2). \end{pmatrix}$$

An even lengthier calculation using the residue cocycle from the local index theorem, [CPRS2, CPRS3], shows that $P(\phi, \theta)$ has pairing with

$$\mathrm{Dirac}_{\mathbf{T}^2} = \left(C^\infty(\mathbf{T}^2), L^2(\mathbf{T}^2) \otimes \mathbf{C}^2, \begin{pmatrix} 0 & -\partial_\phi + i\partial_\theta \\ \partial_\phi + i\partial_\theta & \end{pmatrix} \right)$$

equal to one. Hence p is the desired generator.

Example. Consider the 2-graph Λ_n whose skeleton is illustrated in Figure 4
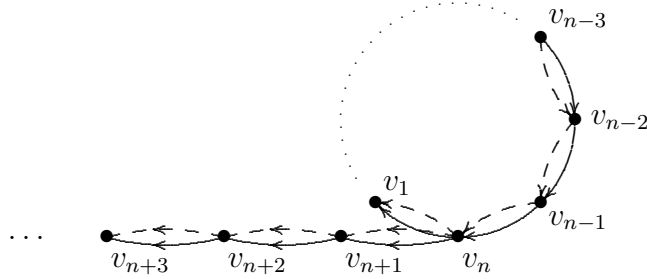

FIGURE 4. The 2-graph Λ_n

We label the solid edge whose range is v_i by e_i and the dashed edge with the same range is labelled f_i for all $i < n$. Without the infinite tail to the left, we think of this 2-graph as the 'n-point 2-torus'. To justify this, note that Λ_n is the pull-back of the graph E_n from [PRen] with respect to the functor $\lambda \mapsto d(\lambda)_1 + d(\lambda)_2$, and so [KP, Corollary 3.5(iii)] shows that

$$C^*(\Lambda_n) \cong C^*(E_n) \otimes C(\mathbf{T}) \cong \mathcal{K} \otimes C(\mathbf{T}) \otimes C(\mathbf{T}) \cong \mathcal{K} \otimes C(\mathbf{T}^2);$$

in particular p_{v_n} is a full projection in $C^*(\Lambda_n)$ and $p_{v_n} C^*(\Lambda_n) p_{v_n}$ is isomorphic to $C(\mathbf{T}^2)$.

We wish to describe the isomorphism explicitly. To do this, first notice that $p_{v_n} C^*(\Lambda) p_{v_n}$ is generated by the unitaries

$$u_1 := s_{e_1} s_{e_2} \dots s_{e_n} \quad \text{and} \quad u_2 = s_{f_1} s_{e_2} \dots s_{e_n}.^1$$

For $1 \le i < j$ let $\theta_{i,j} := s_{e_j} \dots s_{e_{i+1}}$, for $i > j$, let $\theta_{i,j} := \theta_{j,i}^* = s_{e_{j+1}}^* \dots s_{e_i}^*$, and for $i = j$, let $\theta_{i,i} := p_{v_i}$. For $i = 1, 2$ define $U_i \in \mathcal{M}(C^*(\Lambda_n))$ by $U_i := \sum_{j \in \mathbf{N}} \theta_{n,j} u_i \theta_{j,n}$.

---

[1]Note that while $w = s_{f_1} s_{f_2} \cdots s_{f_n}$ would appear to be a more natural candidate for the second generator than u_2, it is easy to see that u_2 does not belong to $C^*(\{u_1, w\})$ and so u_1 and w do not generate $p_{v_n} C^*(\Lambda) p_{v_n}$. From a K-theoretic point of view, however, the distinction is not important: one can check that u_2 and w have the same class in $K_1(p_{v_n} C^*(\Lambda) p_{v_n})$.

Lemma 7.2. *The C^*-algebra $C^*(\Lambda_n)$ is generated by the elements $\{\theta_{i,j} : i, j \in \mathbf{N}\}$ together with $\{U_1, U_2\}$. The $\theta_{i,j}$ form a system of nonzero matrix units, the unitaries U_1, U_2 commute and have full spectrum, and hence there is an isomorphism of $C^*(\Lambda_n)$ onto $\mathcal{K} \otimes C(\mathbf{T}^2)$ which takes $\theta_{i,j} U_1^m U_2^n$ to the function $(w, z) \mapsto \Theta_{i,j} \otimes w^m z^n$. Moreover, the core $F_n = C^*(\Lambda_n)^\gamma$ is isomorphic to $\oplus_{j=1}^n \mathcal{K}$.*

Proof. The Cuntz-Kreiger relations show that the $\theta_{i,j}$ are matrix units and that U_1 and U_2 are unitaries. Since the $\theta_{n,j}$ have orthogonal range projections, $C^*(\{U_1, U_2\})$ is canonically isomorphic to $C^*(\{u_1, u_2\})$, which in turn is canonically isomorphic to $C(\mathbf{T}^2)$ (see Proposition A.6). It is easy to check that U_1 and U_2 commute with the matrix units $\theta_{i,j}$ so $C^*(\{U_1, U_2, \theta_{i,j} : i, j \in \mathbf{N}\}) \cong \mathcal{K} \otimes C(\mathbf{T}^2)$ (it is worth noting that compression by $p_{v_n} = \theta_{n,n}$ takes U_i to u_i). It now remains to show that this algebra is all of $C^*(\Lambda_n)$.

For $i \neq 1$, we have $s_{e_i} = \theta_{i,i+1}$, and we have $s_{e_1} = u_1 \theta_{n,1}$ and $s_{f_2} = u_2 \theta_{n,1}$. The only possible factorisation rule for Λ_n satisfies $e_{i+1} f_i = f_{i+1} e_i$ for all i, and it now follows that $s_{f_i} = \theta_{1,i} s_{f_1} \theta_{i-1,n}$ for all $i > 1$. Hence all the generators of $C^*(\Lambda_n)$ belong to $C^*(\{U_1, U_2, \theta_{i,j} : i, j \in \mathbf{N}\})$ and it follows that $C^*(\Lambda_n)$ is isomorphic to $\mathcal{K} \otimes C(\mathbf{T}^2)$ as required.

To see that F_n is isomorphic to $\oplus_{i=1}^n \mathcal{K}$, first observe that the subalgebra $C^*(\{s_\alpha : \alpha \in \Lambda_n : d(\alpha)_2 = 0\})$ generated by paths consisting only of solid edges is canonically isomorphic to the graph algebra $C^*(E_n)$ described in [PRen, Corollary 6.6], and that this isomorphism intertwines the restriction of the gauge action on $C^*(\Lambda_n)$ to $(\mathbf{T}, 1)$ and the gauge action on $C^*(E_n)$.

It is shown in [PRen] that the core of $C^*(E_n)$ is isomorphic to $\oplus_{i=1}^n \mathcal{K}$: the minimal projections in the jth copy of $\mathcal{K}$ are the vertex projections $s_{v_i} : i \cong l \bmod n$, and for $i \geq j$, the $(i, j)^{\text{th}}$ matrix unit is $\theta_{i,j}^l := s_\eta s_{L(v_j)^{i-j}}^* s_\zeta^*$ where η is the shortest path from v_l to v_{in+l}, ζ is the shortest path from v_l to v_{jn+l}, and $L(v_l)$ is the loop of length n based at v_l.

Hence it suffices to show here that

(12) $$F_n = \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu), d(\mu)_2 = d(\nu)_2 = 0, s(\mu) = s(\nu)\}.$$

Recall from [RSY, Section 4.1] that

$$F_n = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda_n, d(\mu) = d(\nu), s(\mu) = s(\nu)\},$$

so we just need to show that if $\mu, \nu \in \Lambda_n$ satisfy $d(\mu) = d(\nu)$ and $s(\mu) = s(\nu)$, then there exist η and ζ such that $d(\eta) = d(\zeta) = (c, 0)$ for some $c \in \mathbf{N}$, $s(\eta) = s(\zeta)$, and $s_\eta s_\zeta^* = s_\mu s_\nu^*$. Fix $\mu, \nu \in \Lambda_n$ with $d(\mu) = d(\nu)$ and $s(\mu) = s(\nu)$, and write (a, b) for $d(\mu)$. Let β be the unique path of degree $(b, 0)$ whose range is equal to the source of μ. By the factorisation property we can express $\mu = \mu_1 \mu_2$ and $\nu = \nu_1 \nu_2$ where $d(\mu_1) = d(\nu_1) = (a, 0)$ and $d(\mu_2) = d(\nu_2) = (0, b)$. Applying the factorisation property again, we obtain

$$\mu\beta = \mu_1 \mu_2 \beta = \mu_1 \beta' \mu_2' \quad \text{and} \quad \nu\beta = \nu_1 \nu_2 \beta = \nu_1 \beta'' \nu_2'$$

where $d(\beta') = d(\beta'') = (b, 0)$ and $d(\mu_2') = d(\nu_2') = (0, b)$. Since $|\mu_1 \beta'| = |\mu_1| + |\beta| = a + b = |\mu|$, we have $s(\mu_1 \beta') = s(\mu)$, and similarly $s(\nu_1 \beta'') = s(\nu) = s(\mu)$. Hence μ_2' and ν_2' are two paths with the same degree and same range. Since $v\Lambda_n^p$ is a singleton for each v and p, it follows that

$\mu_2' = \nu_2'$, so $s_{\mu_2'}s_{\nu_2'}^* = p_{s(\mu)}$ by (CK4). But now

$$s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* = s_\mu s_\beta s_\beta^* s_\nu^* = s_{\mu_1\beta'\mu_2'} s_{\nu_1\beta''\nu_2'}^* = s_{\mu_1\beta'} s_{\mu_2'} s_{\mu_2'}^* s_{\nu_1\beta''} = s_{\mu_1\beta'} s_{\nu_1\beta''}.$$

Since $d(\mu_1\beta') = (a+b, 0) = d(\nu_1\beta'')$ and since $s(\beta') = s(\beta'') = s(\mu)$, this establishes (12). $\qquad\square$

So for all n we have $K_1(C^*(\Lambda_n)) \cong K_0(C^*(\Lambda_n)) \cong \mathbf{Z}^2$. Choose any unitaries $v_1, v_2 \in (C^*(\Lambda_n)_c)^+$, the one-point unitization of the span of the generators, such that v_1, v_2 represent the classes of the standard generators z_1, z_2 of $K_1(C(\mathbf{T}^2))$. Then we obtain, as above, a projection $P(v_1, v_2)$ representing the class of the Bott generator in $K_0(C^*(\Lambda_n))$. Using this, we may compute the pairing of the Kasparov module $(X_n, \mathcal{D}_n)$ constructed for $C^*(\Lambda_n)$ with the Bott projector.

As in [PRen] we will compute first with the 'n-point 2-torus', the analogous calculation for the 2-graph Λ_n will then follow from the isomorphism $K_0(\mathcal{K}^n) \cong K_0(\mathbf{C}^n) = \mathbf{Z}^n$.

Let $\phi : C(\mathbf{T}^2) \to M_n(C(\mathbf{T}^2))$ be given by

$$\phi(f(z_1, z_2)) = \theta_{n,n} f(w_1, w_2) \theta_{n,n} + (1 - \theta_{n,n}) = p_{v_n} f(w_1, w_2) p_{v_n} + (1 - p_{v_n}).$$

Here we have set $w_1 = u_1$ and $w_2 = w$, as in the proof of Lemma 7.2, and denoted the generating unitaries of $C(\mathbf{T}^2)$ by z_1, z_2. Also $\theta_{n,n}$ is the projection p_{v_n}. Let $(X, \mathcal{D})$ be the Kasparov module for the n-point 2-torus built from the gauge action of $\mathbf{T}^2$. Then $\mathcal{D} = \sum_{j=1}^n p_{v_j} \mathcal{D} = \sum p_{v_j} \mathcal{D} p_{v_j}$, and the pull-back of $(X, \mathcal{D})$ by ϕ is

$$\phi^*(X, \mathcal{D}) = (p_{v_n} X, p_{v_n} \mathcal{D}) \oplus \text{ degenerate module} \in KK^0(C(\mathbf{T}^2), F)$$

since $1 - p_{v_n}$ commutes with $\mathcal{D}$. The isomorphism $\psi : F \to \mathbf{C}^n$ given by

$$\psi(\sum_{j=1}^n z_j p_{v_j}) = (z_1, \ldots, z_n)$$

gives us

$$\psi_* \phi^*(X, \mathcal{D}) = \oplus_{j=1}^n (p_{v_n} X p_{v_j}, p_{v_n} \mathcal{D}) \in \oplus_{j=1}^n K^0(C(\mathbf{T}^2).$$

The class of $(p_{v_n} X p_{v_j}, p_{v_n} \mathcal{D})$ is easily seen to be the Dirac operator on $\mathbf{T}^2$ for the usual flat metric. In the following we will identify F with $\mathbf{C}^n$ (suppressing ψ).

Now we can compute the pairing of $\mathcal{D}$ with $P(w_1, w_2) = \phi(P(z_1, z_2))$ where $P(z_1, z_2)$ is the Bott projector of $\mathbf{T}^2$ constructed earlier. We have

$$
\begin{aligned}
\langle [P(w_1, w_2)], [(X_n, \mathcal{D}_n)] \rangle &= \langle \phi_*([P(z_1, z_2)], [(X_n, \mathcal{D}_n)] \rangle \\
&= \langle [P(z_1, z_2)], \phi^*[(X_n, \mathcal{D}_n)] \rangle \text{ functoriality of Kasparov product} \\
&= \langle [P(z_1, z_2)], [(p_{v_n} X_n, p_{v_n} \mathcal{D}_n)] \oplus [\text{degenerate module}] \rangle \\
&= \langle P(z_1, z_2), \oplus_{j=1}^n [(p_{v_n} X_n p_{v_j}, p_{v_n} \mathcal{D}_n)] \rangle \\
&= \oplus_{j=1}^n \langle P(z_1, z_2), [\text{Dirac}_{\mathbf{T}^2}] \rangle \\
&= \oplus_{j=1}^n \langle [z_1] \times [z_2], [\text{Dirac}_{\mathbf{T}^1}] \times [\text{Dirac}_{\mathbf{T}^1}] \rangle \text{ by [HR, Theorem 10.8.7]} \\
&= - \oplus_{j=1}^n \langle [z_1], [\text{Dirac}_{\mathbf{T}^1}] \rangle \langle [z_2], [\text{Dirac}_{\mathbf{T}^1}] \rangle \text{ [HR, Chapter 9]} \\
&= -(1, 1, \ldots, 1) \in \mathbf{Z}^n = K_0(\mathbf{C}^n).
\end{aligned}
$$

(13)

Using Theorem 7.1, we may compute the pairing of the Bott class with the spectral triple $(\mathcal{A}, \mathcal{H}_n, \mathcal{D}_n)$ (where the k-graph trace is chosen to be equal to 1 on each vertex) by applying $\tilde{\tau}_*$ to this last computation. We obtain

$$\langle [P(w_1, w_2)], [(\mathcal{A}, \mathcal{H}_n, \mathcal{D}_n)] \rangle = -n.$$

The number n appears basically because the multiplicity provided by the core has given us n copies of the Dirac operator at each point.

Now one can add the handle to the n-point 2-torus to get the 2-graph Λ_n. The core becomes $\mathcal{K}^n$ and an argument entirely analogous to the above shows again that the number n emerges from the pairing of $\mathcal{D}$ with the class of the Bott projector.

Note that this example can be generalised to an n-point k-torus with a "handle". A similar argument to that of Lemma 7.2 shows that the resulting k-graph Λ_n^k satisfies $C^*(\Lambda_n^k) \cong \mathcal{K} \otimes C(\mathbf{T}^k)$ independent of n, but that the core F_n^k is always isomorphic to $\oplus_{i=1}^n \mathcal{K}$. We can therefore see that n appears in the index computation in each case.

The point of this example is as follows. Whilst $C^*(\Lambda_n) \cong C^*(\Lambda_m)$ for all n, m, as k-graph algebras they are, somewhat vaguely, 'different' for $n \neq m$. This difference is embodied by equivalently the different gauge actions, the nonisomorphic cores, and the different presentations as universal algebras. Because $(X_n, \mathcal{D}_n)$ is constructed from the gauge action, one would expect that $[(X_n, \mathcal{D}_n)] \in KK^0(C^*(\Lambda_n), C^*(\Lambda_n)^\gamma)$ could 'see' these differences.

However, given a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(\mathcal{N}, \tau)$, we have no knowledge of the possible range of the index; any real number is possible _a priori_. What theorem 7.1 says, at least in this case, is that the semifinite index _is_ 'quantised' — the resulting index for $C^*(\Lambda_n)$ is always a multiple of n. In [KNR], inspired by the result for k-graphs, it is shown that a similar result is true for any semifinite spectral triple.

We list two results for this very simple example which indicate the kinds of information one may draw from semifinite index theory in general:

1) No combination of operator homotopy and addition of degenerate spectral triples, [CPRS1], can make $(A_c, \mathcal{H}_n, \mathcal{D}_n)$ and $(A_c, \mathcal{H}_m, \mathcal{D}_m)$ unitarily equivalent.

2) The gauge actions on $C(\mathbf{T}^2) \otimes \mathcal{K}$ coming from the presentations as $C^*(\Lambda_n)$ and $C^*(\Lambda_m)$, $n \neq m$, are not homotopic in $\mathrm{Aut}(C(T^2) \otimes K)$.

From the point of view of k-graph algebas, what is interesting here is not the differences between Λ_m and Λ_n, or between the cores of the corresponding C^*-algebras, but rather that the semifinite index can detect these differences in the algebras. For example, while it is obvious that for $n \neq m$ there is no gauge equivariant isomorphism $\phi : C^*(\Lambda_n) \to C^*(\Lambda_m)$ (such a map would give an isomorphism $\mathcal{K}^n \cong \mathcal{K}^m$ of the fixed point algebras), and hence there is no functorial isomorphism between the 2-graphs Λ_n and Λ_m (such an equivalence would give rise to a gauge equivariant isomorphism of C^*-algebras), it is not obvious that the semifinite index (which sees the C^*-algebra and not the graph) should detect such information.

Thus the semifinite index reflects finer details of the C^*-algebra than the ordinary Fredholm index possibly could.

Appendix A. k-graphs which admit faithful graph traces

In this appendix we formulate two necessary conditions (Lemma A.1 and Corollary A.2), and one sufficient condition (Proposition A.3), for a k-graph Λ to admit a faithful graph trace. Our sufficient condition is certainly much stronger than need be; indeed, the C^*-algebra of a k-graph satisfying our condition is Morita equivalent to a commutative C^*-algebra whereas, for example, [PRRS] contains many examples of 2-graphs which admit faithful graph traces and whose C^*-algebras are simple A**T** algebras with real rank 0. However, our condition is a direct generalisation of the corresponding result in [PRen] which has already attracted independent interest. Moreover, the results about k-graphs satisfying this condition, including the K-theory calculations, are new and should be of independent interest to the k-graph community.

For the purposes of our first two results we say that paths μ and ν in a k-graph Λ are *orthogonal* if the range projections $s_\mu s_\mu^*$ and $s_\nu s_\nu^*$ are orthogonal in $C^*(\Lambda)$. By [RSY, Proposition 3.5], μ and ν are orthogonal if and only if they have no common extensions.

Lemma A.1 (cf [PRen, Lemma 3.5]). *Suppose that (Λ, d) is a row-finite k-graph and there are vertices $v, w \in \Lambda^0$ with an infinite number of mutually orthogonal paths from w to v. Then there is no faithful k-graph trace on Λ^0.*

Proof. Let $(\lambda_n)_{n \in \mathbf{N}}$ be the infinite set of orthogonal paths from w to v. Suppose that τ is a trace on $C^*(\Lambda)$. For each n, $\tau(s_{\lambda_n} s_{\lambda_n}^*) = \tau(s_{\lambda_n}^* s_{\lambda_n}) = \tau(p_w)$. It follows that for any N, we have $\tau(p_v) \geq \sum_{n=1}^N \tau(s_{\lambda_n} s_{\lambda_n}^*) = N\tau(p_w)$, and it follows that $\tau(p_w) = 0$. Hence $g_\tau(w) = 0$, and it follows from Proposition 3.8 that no k-graph trace on Λ^0 is faithful. $\qquad\square$

Corollary A.2 (cf [PRen, Corollary 3.7]). *Suppose that (Λ, d) is a row-finite k-graph and there exists a vertex $v \in \Lambda^0$ with an infinite number of mutually orthogonal paths from an end to v. Then there is no faithful k-graph trace on Λ^0.*

Proof. Since k-graph traces are constant on ends by Remarks 3.6, the proof is identical to that of Lemma A.1. $\qquad\square$

We now aim to provide a sufficient condition for a k-graph to admit a faithful k-graph trace.

Notation. Let Λ be a locally convex row-finite k-graph. For ends x and y of Λ, we write $x \sim y$ if and only if $x(n) = y(m)$ for some $n \leq d(x)$ and $m \leq d(y)$. This defines an equivalence relation on ends of Λ, and we write $[x]$ for the equivalence class of an end x under $\sim$.

If a vertex v lies on an end of Λ, then $v\Lambda^{\leq\infty} = \{x_v\}$, where x_v is itself an end of Λ.

Proposition A.3 (cf [PRen, Propositions 3.8 and 3.9]). *Let Λ be a locally convex row-finite k-graph. Suppose that there is a function $v \mapsto n_v$ from Λ^0 to $\mathbf{N}^k$ such that for each $v \in \Lambda^0$ and each $\lambda \in v\Lambda^{\leq n_v}$, $s(\lambda)$ lies on an end of Λ.*

(a) *If $g : \Lambda^0 \to \mathbf{R}^+$ is a k-graph trace, then there is a well-defined function from $\mathrm{Ends}(\Lambda)/\sim$ to $\mathbf{R}^+$ satisfying $g([x]) := g(x(0))$, and*

$$(14) \qquad g(v) = \sum_{\lambda \in v\Lambda^{\leq n_v}} g([x_{s(\lambda)}]) \quad \text{for every } v \in \Lambda^0.$$

(b) *Conversely, given any function g from $\mathrm{Ends}(\Lambda)/\sim$ to $\mathbf{R}^+$, there is a unique graph-trace $\overline{g}$ on Λ satisfying $\overline{g}(x(0)) = g([x])$ for all $x \in \mathrm{Ends}(\Lambda)$.*

Before proving the Proposition we need to know that for a fixed function g from $\mathrm{Ends}(\Lambda)/\sim$ to $\mathbf{R}^+$, the formula (14) is independent of the choice of function $v \mapsto n_v$.

Lemma A.4. *Suppose that Λ satisfies the hypotheses of Propopoition A.3, and let g be a function from $\mathrm{Ends}(\Lambda)/\sim$ to $\mathbf{R}^+$. Define $g(v) := g([x_v])$ for each vertex v that lies on an end of Λ. Fix $v \in \Lambda^0$ and suppose $n_1, n_2 \in \mathbf{N}^k$ each have the property that $s(\lambda)$ lies on an end of Λ for each $\lambda \in v\Lambda^{\leq n_i}$. Then*

$$\sum_{\mu \in v\Lambda^{\leq n_1}} g(s(\mu)) = \sum_{\nu \in v\Lambda^{\leq n_2}} g(s(\nu)).$$

Proof. Let $n := n_1 \vee n_2$. Using that $v\Lambda^{\leq n - n_i}$ is a singleton ($i = 1, 2$) when v lies on an end, one easily checks that $\sum_{\mu \in v\Lambda^{\leq n_i}} g(s(\mu)) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda))$ for $i = 1, 2$. $\qquad \square$

Proof of Proposition A.3. (a) By definition, graph traces are constant on ends, and hence on equivalence classes of ends. The formula (14) holds by definition of a k-graph trace.

(b) Define $\overline{g}(v) : \Lambda^0 \to \mathbf{R}^+$ by

$$\overline{g}(v) := \sum_{\lambda \in v\Lambda^{\leq n_v}} g([x_{s(\lambda)}])$$

Note that if x is an end of Λ, then $n_{x(0)} := 0$ has the property that $s(\lambda)$ lies on an end of Λ for each $\lambda \in x(0)\Lambda^{n_{x(0)}}$. Hence Lemma A.4 shows that $\overline{g}(x(0)) = g([x])$.

Fix $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We must show that

$$(15) \qquad \overline{g}(v) = \sum_{\lambda \in v\Lambda^{\leq n}} \overline{g}(s(\lambda)).$$

We may assume without loss of generality that $n \leq n_v$ because if it is not, then $n'_v := n \vee n_v$ can be used in place of n_v by Lemma A.4 and satisfies $n \leq n'_v$. Since $\Lambda^{\leq n_v} = \Lambda^{\leq n}\Lambda^{\leq n_v - n}$ [RSY, Lemma 3.6], we then have that for each $\lambda \in v\Lambda^{\leq n}$, the element $n_v - n$ has the property that for each $\alpha \in s(\lambda)\Lambda^{\leq n_v - n}$, the source of α is on an end of Λ and hence

$$(16) \qquad \overline{g}(s(\lambda)) = \sum_{\alpha \in s(\lambda)\Lambda^{\leq n_v - n}} g(s(\alpha))$$

by Lemma A.4. But now

$$
\begin{aligned}
\sum_{\lambda \in v\Lambda^{\leq n}} \overline{g}(s(\lambda)) &= \sum_{\lambda \in v\Lambda^{\leq n}} \Big(\sum_{\alpha \in s(\lambda)\Lambda^{\leq n_v - n}} g(s(\alpha)) \Big) \quad \text{by (16)} \\
&= \sum_{\lambda\alpha \in v\Lambda^{\leq n_v}} g(s(\lambda\alpha)) \quad \text{by [RSY, Lemma 3.6]} \\
&= \overline{g}(v)
\end{aligned}
$$

by definition of $\overline{g}$. $\square$

Finally we show that we can check that a given function is a graph trace just by considering edges and vertices in the skeleton of Λ. This is useful as it simplifies the task of checking that a given function is a k-graph trace.

Lemma A.5. *Let Λ be a locally-convex row-finite k-graph. Suppose that $g : \Lambda^0 \to \mathbf{R}^+$ satisfies $g(v) = \sum_{e \in v\Lambda^{e_i}} g(s(e))$ for all $v \in \Lambda^0$ and all $1 \leq i \leq k$ such that $v\Lambda^{e_i} \neq \emptyset$. Then g is a k-graph trace.*

Proof. We proceed by induction on $\ell(n) = \sum_{i=1}^k n_i$. If $\ell(n) = 0$ then $n = 0$ and $v\Lambda^{\leq n} = \{v\}$ for each v, so (2) holds trivially. Suppose as an inductive hypothesis that (2) holds for $\ell(n) \leq L$, fix $v \in \Lambda^0$ and $n \in \mathbf{N}^k$ with $\ell(n) = L + 1$, and write $n = n' + e_i$ where $\ell(n') = L$. By the inductive hypothesis,

$$
(17) \qquad\qquad g(v) = \sum_{\lambda \in v\Lambda^{\leq n'}} g(s(\lambda)),
$$

and by hypothesis, we know that for each $\lambda \in v\Lambda^{n'}$, we have $g(s(\lambda)) = \sum_{e \in s(\lambda)\Lambda^{\leq e_i}} g(s(e))$ (if $s(\lambda)\Lambda^{\leq e_i} = \{s(\lambda)\}$, this is trivial, and otherwise it is precisely the hypothesis of the lemma. Since [RSY, Lemma 3.6] ensures that $(\lambda, e) \mapsto \lambda e$ is a bijection from $\{(\lambda, e) : \lambda \in v\Lambda^{\leq n'}, e \in s(\lambda)\Lambda^{\leq e_i}\}$ to $v\Lambda^{\leq n}$, a straightforward calculation shows that $g(v) = \sum_{\mu \in v\Lambda^{\leq n}} g(s(\mu))$. $\square$

A.1. **The C^*-algebras of k-graphs which admit k-graph traces.** In this subsection we give some structural results and K-theory calculations for $C^*(\Lambda)$ when Λ is a k-graph which satisfies the hypotheses of Proposition A.3.

Proposition A.6. *Let Λ be a k-graph, and suppose that the boundary path $x : \Omega_{k,m} \to \Lambda$ is surjective. Let v denote the vertex $x(0) \in \Lambda^0$. Then*

 (a) *the collection $G := \{p - q : p, q \leq d(x), x(p) = x(q)\}$ is a subgroup of $\mathbf{Z}^k$;*
 (b) *the projection p_v is full in $C^*(\Lambda)$; and*
 (c) *there is an isomorphism ϕ of the full corner $p_v C^*(\Lambda) p_v$ onto the subalgebra $C^*(\{L_n : n \in G\}) \subset C^*(\mathbf{Z}^k)$ which satisfies $\phi(s_{x(0,p)} s^*_{x(0,q)}) = L_{p-q}$ whenever $x(p) = x(q)$.*

In particular, $C^*(\Lambda)$ is Morita equivalent to $C^*(G)$ which is isomorphic to $C(\mathbf{T}^l)$ for some $0 \leq l \leq k$.

Proof. (a) G clearly contains the identity, and is closed under inverses by symmetry. Suppose that $x(p) = x(q)$ and $x(p') = x(q')$, so $n = p - q$ and $n' = p' - q'$ belong to G. We must show that $n + n' \in G$. Since $q, p' \le d(x)$, we have $q \vee p' \le d(x)$. Let $\alpha := x(q, q \vee p')$ and let $\beta := x(p', q \vee p')$. Clearly $s(\alpha) = s(\beta)$. But $r(\alpha) = x(q) = x(p)$, and since x is surjective, it follows that $x(0, p)\alpha = x(0, p + (q \vee p') - q)$; and similarly, we have $x(0, q')\beta = x(0, q' + (q \vee p') - p')$. Hence $x(p + (q \vee p') - q) = x(q' + (q \vee p') - p')$, and so $n + n' = p - q + p' - q' = (p + (q \vee p') - q) - (q' + (q \vee p') - p')$ belongs to G as required.

(b) Since x is surjective, the hereditary subset of $C^*(\Lambda)$ generated by v as in [RSY, §5] is all of Λ^0, so the ideal generated by p_v is $C^*(\Lambda)$ as required.

(c) We have that $p_v C^*(\Lambda) p_v = \overline{\text{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in v\Lambda, s(\lambda) = s(\mu)\} = \overline{\text{span}}\{s_{x(0,p)} s_{x(0,q)}^* : x(p) = x(q)\}$ because x is surjective. Suppose that $x(p) = x(q)$ and $x(p') = x(q')$, and that $p - q = p' - q'$. Using (CK4) and that x is surjective, one checks that $s_{x(0,p)} s_{x(0,q)}^* = s_{x(0,p')} s_{x(0,q')}^*$, so for $n \in G$ we may define $U_n := s_{x(0,p)} s_{x(0,q)}^*$ for any p, q such that $x(p) = x(q)$ and $p - q = n$.

The Cuntz-Krieger relations show that $U_n U_n^* = U_n^* U_n = p_v$ for all n, so the U_n are unitaries.

The calculation
$$U_{q-p} = s_{x(0,q)} s_{x(0,p)}^* = (s_{x(0,p)} s_{x(0,q)}^*)^* = U_{p-q}^*.$$
Shows that $U_{-n} = U_n^*$ for all $n \in G$. Moreover, since x is surjective, we have $s_{x(0,q)}^* s_{x(0,p')} = s_{x(q, q \vee p')} s_{x(p', q \vee p')}^*$ by [RSY, Proposition 3.5] and (CK3), and it follows that
$$U_{p-q} U_{p'-q'} = s_{x(0,p)} s_{x(0,q)}^* s_{x(0,p')} s_{x(0,q')}^* = s_{x(0, p+(q \vee p')-q)} s_{x(0, q'+(q \vee p')-p')}^* = U_{p+p'-q-q'}$$
so that $n \mapsto U_n$ is a representation of G. It follows that there is a surjective C^*-homomorphism $\phi : C^*(G) \to C^*(\{U_n : n \in G\}) = p_v C^*(\Lambda) p_v$ which satisfies $\phi(\chi_n) = U_n$ for all $n \in G$.

It remains only to show that ϕ is injective. For this, we need only show that $U_m \ne U_n$ for $m \ne n$ and that each U_n where $n \ne 0$ has full spectrum. Since $\gamma_z(U_m U_n^*) = z^{m-n} U_m U_n^*$, an appropriate choice of $z \in \mathbf{T}^k$ shows that $U_m \ne U_n$ for $m \ne n$. That each U_n has full spectrum follows from an argument identical to that used in [PRRS, Lemma 3.9]. This establishes (c).

The Morita equivalence of $C^*(\Lambda)$ with $C^*(G)$ follows immediately from (2) and (3), and since G is a subgroup of $\mathbf{Z}^k$, we must have $G \cong \mathbf{Z}^l$ for some $0 \le l \le k$. $\square$

For the remainder of the appendix, Λ will be a fixed k-graph which satisfies the hypotheses of Proposition A.3.

A vertex $v \in \Lambda^0$ lies on an end of Λ if and only if $|v\Lambda^{\le n}| = 1$ for all $n \in \mathbf{N}^k$. Let $\text{Ends}(\Lambda)^0$ denote the collection of all such vertices; for each $v \in \text{Ends}(\Lambda)^0$, there is a unique end $x(v)$ whose range is v. By a simple argument, we may select a set $V \subset \text{Ends}(\Lambda)^0$ such that for each $x \in \text{Ends}(\Lambda)$ there is a unique $v \in V$ such that $x \sim x(v)$. We fix this collection for the remainder of the section.

Proposition A.7. *For each $v \in V$, let $\Lambda(v)$ be the image of $x(v)$ which is a subcategory of Λ. Then each $(\Lambda(v), d|_{\Lambda(v)})$ is itself a k-graph, and $C^*(\Lambda)$ is Morita equivalent to $\bigoplus_{v \in V} C^*(\Lambda(v))$.*

Proof. Since each $x(v)$ is an end, each $\Lambda(v)^0$ is a hereditary subset of Λ^0. For distinct $v, w \in V$, we have $\Lambda(v) \cap \Lambda(w) = \emptyset$ because otherwise $x(v) \sim x(w)$ contradicting our choice of V. By [RSY, Theorem 5.2], for each $v \in \Lambda^0$, the projection $P_v := \sum_{w \in \Lambda(v)^0} p_w$ determines an ideal $I_v := C^*(\Lambda)P_v C^*(\Lambda)$ which is Morita equivalent to $C^*(\Lambda(v)^0\Lambda) = C^*(\Lambda(v))$. Each $I_v = \overline{\text{span}}\{s_\lambda s_\mu^* : s(\lambda) = s(\mu) \in \Lambda(v)^0\}$, and since the distinct $\Lambda(v)$ do not intersect the ideal generated by all the P_v is isomorphic to $\bigoplus_{v \in V} I_v$.

The assumption that for each vertex $w \in \Lambda^0$ there is an element $n_w \in \mathbf{N}^k$ such that $s(w\Lambda^{\leq n_w}) \subset \text{Ends}(\Lambda)^0$ guarantees that every vertex of Λ belongs to the saturated hereditary set generated by V. Another application of [RSY, Theorem 5.2] shows that the ideal generated by all the I_v is $C^*(\Lambda)$. Hence $C^*(\Lambda) = \bigoplus I_v$ is Morita equivalent to $\bigoplus C^*(\Lambda(v))$. $\qquad\square$

Corollary A.8. *For each $v \in V$, let $G_v := \{p - q : x(v)(p) = x(v)(q)\} \subset \mathbf{Z}^k$. Then $C^*(\Lambda)$ is Morita equivalent to $\bigoplus_{v \in V} C^*(G_v) \cong \bigoplus_{v \in V} C(\mathbf{T}^{l_v})$ where $0 \leq l_v \leq k$ for each v. In particular $K_*(C^*(\Lambda))$ is isomorphic to $\bigoplus_{v \in V} K_*(C(\mathbf{T}^{l_v}))$.*

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